

Quantum quench in the sine-Gordon model

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We consider the time evolution in the repulsive sine-Gordon quantum field theory after the system is prepared in a particular class of initial states. We focus on the time dependence of the one-point function of the semi-local operator $\exp(i\beta\Phi(x)/2)$. By using two different methods based on form-factor expansions, we show that this expectation value decays to zero exponentially, and we determine the decay rate by analytical means. Our methods generalize to other correlation functions and integrable models.

I. INTRODUCTION

The last decade has witnessed great advances in the understanding of the non-equilibrium dynamics of isolated many-particle quantum systems. For example, it has become possible to generate one-dimensional quantum gases and study their time evolution after a quantum quench, i.e. a sudden change in the system parameters^{1,2}. The quantum Newton's cradle¹ in particular emphasized the importance of conservation laws and motivated the study of quantum quenches in integrable models. One key result of this work has been that the stationary properties of observables at late times after the quench can be described in terms of a generalized Gibbs ensemble (GGE)³, which has by now been fully established for several systems^{4,5}, while strong evidence in favour of the GGE has been found for many others⁶⁻⁹. Generalisations of the GGE to initial states in which the probability distributions of conserved charges are not infinitely sharp in the thermodynamic limit are discussed in Ref. [10].

A recent focus of attention has been the problem of determining expectation values of local observables in the stationary state after a quantum quench in *interacting* integrable models, and the construction of appropriate GGEs for these cases¹¹⁻¹³.

The problem of describing the time evolution of local observables in integrable models after a quantum quench is harder and less well developed. So far analytic results have been obtained only for certain quenches to conformal field theories¹⁴ and models that can be mapped to free fermions or bosons^{5,7-9,15-17}. A logical next step is then to develop techniques for describing the time evolution after quenches to interacting integrable models. Here we address this problem for a paradigmatic integrable quantum field theory, the quantum sine-Gordon model

$$\mathcal{H}_{\text{SG}} = \frac{v}{16\pi} \int dx \left[(\partial_x \Phi)^2 + \frac{1}{v^2} (\partial_t \Phi)^2 \right] - \lambda \int dx \cos(\beta\Phi). \quad (1)$$

We are interested in the situation where the system is prepared in a particular kind of initial state $|\Psi_0\rangle$, which is not an eigenstate of \mathcal{H}_{SG} . Here we focus on the case where $|\Psi_0\rangle$ corresponds to an integrable boundary condition in the sine-Gordon model¹⁸. This choice of initial state is motivated by the situation encountered for quenches to conformal field theories¹⁴, which can be mapped to a boundary problem with appropriately defined boundary state. Our protocol of imposing a particular initial state differs from the usual quench setup, where one prepares the system in the ground state of a given Hamiltonian and then changes one or more system parameters instantaneously. We discuss the realizability of integrable boundary states in standard quench protocols in Section IV. Given our initial state, we consider unitary time evolution

$$|\Psi_t\rangle = e^{-i\mathcal{H}_{\text{SG}}t} |\Psi_0\rangle, \quad (2)$$

and our goal is to determine expectation values of the kind

$$F_{\Psi_0}^\alpha(t) = \frac{\langle \Psi_t | e^{i\alpha\Phi(x)} | \Psi_t \rangle}{\langle \Psi_t | \Psi_t \rangle}. \quad (3)$$

We do so by two complementary approaches. The first is based on the “representative state” formalism proposed in Ref. [16]. The second is a generalisation of the low-density expansion introduced for studying quenches in the transverse field Ising chain⁵ and the Ising field theory¹⁷.

Quantum quenches to the massive regime $\beta < 1$ of the sine-Gordon model were considered by Iucci and Cazalilla⁸. They focussed on two special cases, namely quenches from the massless regime to the free-fermion point $\beta = 1/\sqrt{2}$ and in the semiclassical regime $\beta \rightarrow 0$. These cases are characterized by no or weak interactions respectively, and as

a result are amenable to treatment by simpler methods than the ones developed in our work. The solvability of the model at the free-fermion point was also employed by Foster et al.¹⁹ to investigate the amplification of initial density inhomogeneities.

Gritsev et al.²⁰ considered the time evolution of the *local* operator $e^{i\beta\Phi}$ after preparing the system in an integrable boundary state similar to ours. However, they focussed on the attractive regime $\beta < 1/\sqrt{2}$ where, in addition to solitons and antisolitons, breather bound states exist. By employing a form-factor expansion and evaluating the first few terms, they calculated the power spectrum of the vertex operator $e^{i\beta\Phi}$ and showed that it possesses sharp resonances corresponding to the creation of breather states and discussed implications for experiments on split one-dimensional Bose condensates. For the approach Ref. [20] to apply, higher order terms in the Lehmann expansion must be negligible. This is generally not the case^{5,17}: for semi-local operators of the kind considered here these terms are in fact divergent at late times. Our main purpose is to develop methods for determining the late-time behaviour in this much more complicated case.

For $\beta > 1$ the cosine perturbation is irrelevant at low energies and quenches in this very different regime have been considered in Refs. [21].

The remainder of this article is organized as follows: In the next section we provide some context for our work by summarising applications of the sine-Gordon model to magnetic solids and systems of trapped, ultra-cold atoms. This is followed by a brief review of some well-known facts related to the integrability of the sine-Gordon model in section III. We then introduce the notion of “integrable” quantum quenches, and discuss various issues relating to particular kind of initial state we consider. In Section V we determine the time evolution of the one-point function $F_{\Psi_0}^{\beta/2}(t)$ using a method based on the recently proposed “representative state” approach¹⁶. In Section VI we apply a complementary method based on a linked-cluster expansion, which was originally developed for quenches in the transverse-field Ising model^{5,17}. Both approaches show that the one-point function $F_{\Psi_0}^{\beta/2}(t)$ decays exponentially in time, see Eq. (200), with a calculable decay time τ . We conclude with a discussion of our results and an outlook for further investigations in Section VII. Technical details encountered in the course of our analysis are presented in a number of appendices.

II. REALISATIONS OF THE SINE-GORDON MODEL

The sine-Gordon model is known to emerge as the low-energy description in a variety of contexts, both in solids and systems of trapped, ultra-cold atoms.

A. Solids

The sine-Gordon model is obtained as the low-energy limit of a variety of quantum spin chain models, see e.g. Ref. [22]. Perhaps the best experimental realisation in this context is in field-induced gap Heisenberg magnets like CuBenzoate^{23,24}. The underlying lattice model in these systems is of the form

$$H = J \sum_{j=1}^L \left[S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right] + h_u \sum_{j=1}^L S_j^z + h_s \sum_{j=1}^L (-1)^j S_j^x. \quad (4)$$

In the thermodynamic limit the low-energy sector is described by a quantum sine-Gordon model (1), but typically $\beta < \frac{1}{2}$ so that the sine-Gordon model is in the attractive regime. The repulsive regime can be realized as the low-energy limit of the spin-1/2 XYZ chain

$$H = J \sum_{j=1}^L \left[(1 + \gamma) S_j^x S_{j+1}^x + (1 - \gamma) S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right], \quad (5)$$

where we take

$$J > 0, \quad 0 < \Delta < 1, \quad \gamma \ll 1. \quad (6)$$

In this regime we may regard (5) as a perturbation of the spin-1/2 XXZ chain by the relevant operator $\sum_j [S_j^x S_{j+1}^x - S_j^y S_{j+1}^y]$. In the low-energy limit this gives a sine-Gordon model (1) with

$$-\Delta = \cos(\pi\beta^2), \quad \frac{1}{2} < \beta^2 < 1, \quad v = \frac{J \sin(\beta^2)}{2(1 - \beta^2)}, \quad \lambda \propto \gamma. \quad (7)$$

The transverse spin operators are bosonized as follows

$$\begin{aligned} S_j^+ &= S_j^x + iS_j^y \sim \mathcal{C}(-1)^j : e^{i\frac{\beta}{2}\Phi(x)} : + \mathcal{A} \left[: e^{i\frac{\beta}{2}\Phi(x) + \frac{i}{2\beta}\Theta} : + : e^{i\frac{\beta}{2}\Phi(x) - \frac{i}{2\beta}\Theta} : \right] + \dots \\ &\equiv J^+(x) + (-1)^j n^+(x), \quad x = ja_0. \end{aligned} \quad (8)$$

In this realisation of the sine-Gordon model the operator $e^{i\frac{\beta}{2}\Phi(x)}$ thus corresponds to the staggered magnetisation. The bosonized form of the longitudinal spin operator is

$$S_j^z \sim \frac{a_0}{4\pi\beta} \partial_x \Theta(x) - \mathcal{B}(-1)^j \sin\left(\frac{\Theta(x)}{2\beta}\right) + \dots \quad (9)$$

We note that the constants \mathcal{A} , \mathcal{B} and \mathcal{C} are known²⁵. In the XYZ realisation the Bose fields are compactified

$$\Theta(x) = \Theta(x) + 4\pi\beta, \quad \Phi(x) = \Phi(x) + \frac{4\pi}{\beta}. \quad (10)$$

This implies that the only local vertex operators are of the form

$$\mathcal{O}_{n,m} = e^{i\frac{n}{2\beta}\Theta(x) + i\frac{\beta m}{2}\Phi(x)}. \quad (11)$$

Defining the topological charge as

$$Q = \frac{\beta}{2\pi} \int_{-\infty}^{\infty} dx \partial_x \Phi(x), \quad (12)$$

we see that $\mathcal{O}_{n,m}$ carry topological charge $2n$.

B. Systems of ultra-cold trapped atoms

Other realisations of the sine-Gordon model can be found in systems of interacting one dimensional bosons²⁶. For example, one may consider a single species of bosons in a periodic potential

$$H = \frac{1}{2m_0} \int dx |\partial_x \psi|^2 + \frac{1}{2} \int dx dx' V(x-x') \rho(x) \rho(x') + \int dx [V_L(x) - \mu] \rho(x), \quad (13)$$

where $\psi(x)$ is a complex scalar field, $V(x)$ describes density-density interactions between bosons, μ a chemical potential, and $V_L(x)$ a periodic lattice potential. Models like (13) can be realized in systems of trapped, ultra-cold atoms, where $V_L(x)$ accounts for the optical lattice²⁷. “Bosonizing the boson”²⁸

$$\begin{aligned} \psi^\dagger(x) &\sim \sqrt{\rho_0} \sum_{p \in \mathbb{Z}} e^{2ip[\pi\rho_0 x - \sqrt{\frac{K}{8}}\Phi(x)]} e^{-\frac{i}{\sqrt{8K}}\Theta(x)}, \\ \rho(x) &\sim \rho_0 - \sqrt{\frac{K}{8\pi^2}} \partial_x \Phi(x) + \rho_0 \sum_{p \neq 0} e^{2ip[\pi\rho_0 x - \sqrt{\frac{K}{8}}\Phi(x)]}, \end{aligned} \quad (14)$$

where ρ_0 is the average density, leads to a low-energy description²⁶ in terms of a sine-Gordon model (1). Here the $\cos(\beta\Phi)$ originates from the periodic potential $V_L(x)$, and the Luttinger parameter K characterizes the interactions. In order to access the repulsive regime of the sine-Gordon model ($\beta^2 > 1/2$), the density-density interaction $V(x)$ should be sufficiently long-ranged. Of particular interest for our work is the case where the periodic potential is such that there is on average one boson for every two minima of $V_L(x)$, i.e.

$$V_L(x) \sim V_L \cos(4\pi\rho_0 x), \quad (15)$$

which leads to a sine-Gordon model with $\beta^2 = 2K$. As long as $K < 1/2$, the leading oscillating term in the density is

$$\rho_{\text{osc}}(x) \sim \cos\left(2\pi\rho_0 x - \frac{\beta}{2}\Phi(x)\right) + \text{higher harmonics}. \quad (16)$$

Hence in this realisation of the sine-Gordon model our results pertain to the leading oscillatory behaviour of the boson density. We note that the Bose fields again are compactified according to (10).

A second realisation of the sine-Gordon model with ultra-cold atoms is in terms of coupled one-dimensional condensates²⁹. The microscopic Hamiltonian is taken to be

$$H = \sum_{j=1,2} \int dx \left[\frac{1}{2m_0} (\partial_x \Psi_j^\dagger(x)) (\partial_x \Psi_j(x)) + g \rho_j^2(x) \right] - t_\perp \int dx \left(\Psi_1^\dagger(x) \Psi_2(x) + \text{h.c.} \right), \quad (17)$$

where $\rho_j(x) = \Psi_j^\dagger(x) \Psi_j(x)$. Bosonising²⁸ and then transforming to total and relative phases

$$\Phi_\pm(x) = \sqrt{\frac{1}{8K}} \left[\frac{\Phi_1(x) \pm \Phi_2(x)}{\sqrt{2}} \right], \quad \Theta_\pm(x) = \sqrt{\frac{K}{8\pi^2}} \left[\frac{\Theta_1(x) \pm \Theta_2(x)}{\sqrt{2}} \right], \quad (18)$$

one finds a low-energy effective Hamiltonian of the form

$$\begin{aligned} H_+ &= \frac{v}{16\pi} \int dx \left[(\partial_x \Phi_+)^2 + (\partial_x \Theta_+)^2 \right], \\ H_- &= \frac{v}{16\pi} \int dx \left[(\partial_x \Phi_-)^2 + (\partial_x \Theta_-)^2 \right] - \lambda \int dx \cos \left(\frac{\Phi_-}{\sqrt{4K}} \right). \end{aligned} \quad (19)$$

Here $\lambda \propto t_\perp$ and the parameters v and K are functions of g , m_0 and the density ρ_0 of the condensates.

III. SINE-GORDON MODEL AS AN INTEGRABLE QUANTUM FIELD THEORY

In the regime of interest to us here, $1/\sqrt{2} \leq \beta < 1$, the elementary excitations in the sine-Gordon model are massive solitons and antisolitons possessing a relativistic dispersion relation. In the attractive regime $\beta < 1/\sqrt{2}$ there solitons and antisolitons can form bound states known as breathers. At the Luther-Emery point $\beta = 1/\sqrt{2}$ the model is equivalent to a free massive Dirac theory³⁰, thus the quench dynamics can be investigated by simpler methods⁸ than the ones developed here.

A basis of eigenstates of the sine-Gordon model in the repulsive regime is conveniently constructed by employing the Faddeev-Zamolodchikov creation and annihilation operators³¹ $Z_a^\dagger(\theta)$, $Z_a(\theta)$, where the index $a = \pm$ corresponds to the creation and annihilation of solitons and antisolitons respectively. They are taken to fulfil the algebra

$$\begin{aligned} Z_{a_1}(\theta_1) Z_{a_2}(\theta_2) &= S_{a_1 a_2}^{b_1 b_2}(\theta_1 - \theta_2) Z_{b_2}(\theta_2) Z_{b_1}(\theta_1), \\ Z_{a_1}^\dagger(\theta_1) Z_{a_2}^\dagger(\theta_2) &= S_{a_1 a_2}^{b_1 b_2}(\theta_1 - \theta_2) Z_{b_2}^\dagger(\theta_2) Z_{b_1}^\dagger(\theta_1), \\ Z_{a_1}(\theta_1) Z_{a_2}^\dagger(\theta_2) &= 2\pi \delta(\theta_1 - \theta_2) \delta_{a_1, a_2} + S_{a_2 b_1}^{b_2 a_1}(\theta_2 - \theta_1) Z_{b_2}^\dagger(\theta_2) Z_{b_1}(\theta_1), \end{aligned} \quad (20)$$

where $S_{ab}^{cd}(\theta)$ is the two-particle scattering matrix of the sine-Gordon model. For $\beta^2 > \frac{1}{2}$ it is given by

$$\begin{aligned} S_{++}^{++}(\theta) &= S_{--}^{--}(\theta) = S_0(\theta) = -\exp \left[i \int_0^\infty \frac{dt}{t} \sin \left(\frac{t\theta}{\pi\xi} \right) \frac{\sinh \left(\frac{\xi-1}{2\xi} t \right)}{\sinh \left(\frac{t}{2} \right) \cosh \left(\frac{t}{2\xi} \right)} \right], \\ S_{+-}^{+-}(\theta) &= S_{-+}^{-+}(\theta) = S_T(\theta) S_0(\theta), \quad S_T(\theta) = -\frac{\sinh \left(\frac{\theta}{\xi} \right)}{\sinh \left(\frac{\theta - i\pi}{\xi} \right)}, \\ S_{-+}^{+-}(\theta) &= S_{+-}^{-+}(\theta) = S_R(\theta) S_0(\theta), \quad S_R(\theta) = -\frac{i \sin \left(\frac{\pi}{\xi} \right)}{\sinh \left(\frac{\theta - i\pi}{\xi} \right)}, \end{aligned} \quad (21)$$

where we have defined

$$\xi = \frac{\beta^2}{1 - \beta^2}. \quad (22)$$

The S-matrix fulfils the Yang-Baxter equations, crossing and unitarity relations

$$S_{a_2 a_3}^{b_2 b_3}(\theta_2 - \theta_3) S_{a_1 b_3}^{b_1 c_3}(\theta_1 - \theta_3) S_{b_1 b_2}^{c_1 c_2}(\theta_1 - \theta_2) = S_{a_1 a_2}^{b_1 b_2}(\theta_1 - \theta_2) S_{b_1 a_3}^{c_1 b_3}(\theta_1 - \theta_3) S_{b_2 b_3}^{c_2 c_3}(\theta_2 - \theta_3), \quad (23)$$

$$S_{a_1 a_2}^{c_1 c_2}(\theta) S_{c_1 c_2}^{b_1 b_2}(-\theta) = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2}, \quad (24)$$

$$S_{ab}^{cd}(i\pi - \theta) = S_{cb}^{\bar{a}d}(\theta) = S_{ad}^{\bar{c}b}(\theta), \quad \bar{a} = -a. \quad (25)$$

Further relations that will prove useful in the following are

$$(S_{ab}^{cd}(\theta))^* = S_{ab}^{cd}(-\theta) \quad \text{for } \theta \in \mathbb{R}, \quad S_{ab}^{cd}(\theta) = S_{ba}^{dc}(\theta) = S_{cd}^{ab}(\theta) = S_{\bar{a}\bar{b}}^{\bar{c}\bar{d}}(\theta). \quad (26)$$

The zero temperature ground state $|0\rangle$ is defined by

$$Z_a(\theta)|0\rangle = 0, \quad (27)$$

and a basis of eigenstates is obtained by acting with creation operators

$$|\theta_1, \dots, \theta_N\rangle_{a_1 \dots a_N} = Z_{a_1}^\dagger(\theta_1) \dots Z_{a_N}^\dagger(\theta_N)|0\rangle. \quad (28)$$

Energy and momentum of the states (28) are given by

$$E = \Delta \sum_{j=1}^N \cosh \theta_j, \quad P = \frac{\Delta}{v} \sum_{j=1}^N \sinh \theta_j. \quad (29)$$

The topological charge operator Q defined in (12) acts on the basis (28) as

$$Q |\theta_1, \dots, \theta_N\rangle_{a_1 \dots a_N} = \left\{ \sum_{i=1}^N a_i \right\} |\theta_1, \dots, \theta_N\rangle_{a_1 \dots a_N}. \quad (30)$$

Importantly, matrix elements of local operators

$$f^{\mathcal{O}_{a_1 \dots a_N} b_1 \dots b_M}(\theta'_1, \dots, \theta'_M | \theta_1, \dots, \theta_N) = b_1 \dots b_M \langle \theta'_1, \dots, \theta'_M | \mathcal{O}(0) | \theta_1, \dots, \theta_N \rangle_{a_1 \dots a_N} \quad (31)$$

can be calculated by means of the form-factor bootstrap approach^{32–36}. The basic axioms underlying this approach are summarized in Appendix A.

IV. “INTEGRABLE” QUANTUM QUENCHES

As shown by Calabrese and Cardy¹⁴, the calculation of expectation values of local operators after a quench can be mapped to a corresponding problem in boundary quantum field upon analytic continuation to imaginary time. In the case where the post-quench Hamiltonian is integrable, there is then a particular class of boundary conditions, and concomitantly initial states, namely those compatible with integrability¹⁸. Given the relative simplicity of such states, they constitute a natural starting point for quenches to interacting integrable models⁹. A characteristic feature of these states is that they can be cast in the form¹⁸

$$|\Psi_0\rangle = \exp \left(\int_0^\infty \frac{d\theta}{2\pi} K^{ab}(\theta) Z_a^\dagger(-\theta) Z_b^\dagger(\theta) \right) |0\rangle. \quad (32)$$

Here the matrix $K^{ab}(\theta)$ is obtained from a solution of the reflection equations and fulfils

$$K^{a_1 c_1}(\theta_1) K^{c_2 c_3}(\theta_2) S_{c_2 c_1}^{a_2 c_4}(\theta_1 + \theta_2) S_{c_3 c_4}^{b_2 b_1}(\theta_1 - \theta_2) = K^{c_1 b_1}(\theta_1) K^{c_2 c_3}(\theta_2) S_{c_3 c_1}^{b_2 c_4}(\theta_1 + \theta_2) S_{c_2 c_4}^{a_2 a_1}(\theta_1 - \theta_2), \quad (33)$$

$$K^{ab}(\theta) = S_{cd}^{ab}(2\theta) K^{dc}(-\theta). \quad (34)$$

For our purposes it will be useful to exploit the fact that if $K^{ab}(\theta)$ is a solution to (33), other solutions can be obtained by multiplication with an even function $g(\theta)$

$$\tilde{K}^{ab}(\theta) = K^{ab}(\theta) g(\theta). \quad (35)$$

Introducing a function $g(\theta)$ is crucial in the quench context in order to obtain well-defined results (see the discussion below). A particular choice of $g(\theta)$ advocated in Ref. [9] in light of the corresponding situation for conformal field theories¹⁴ is

$$g(\theta) = e^{-2\tau_0 \Delta \cosh \theta}, \quad (36)$$

which defines an “extrapolation time” τ_0 . An immediate question is whether boundary states like (32) make for physically meaningful initial states after quenches to a sine-Gordon model. A particular choice of $K^{ab}(\theta)$ corresponds to fixed boundary conditions

$$\Phi(t = 0, x) = 0. \quad (37)$$

These can be realized by quenching the parameter λ in (1) from infinity to a finite value at time $t = 0$, i.e.

$$\lambda \Big|_{t < 0} = \infty \longrightarrow \lambda \Big|_{t > 0} = \lambda_+. \quad (38)$$

The problem with such a quench is that it corresponds to an initial state after the quench with infinite energy density, which is clearly undesirable. This problem is reflected in the fact that for integrable boundary conditions one typically has

$$\lim_{\theta \rightarrow \infty} \sum_{a,b} |K^{ab}(\theta)|^2 = \text{const}, \quad (39)$$

which essentially corresponds to having a finite density of excitations even for infinite-energy particles. The simplest way to suppress the presence of such excitations in the initial state is to introduce a function $g(\theta)$, and consider an initial state of the form (32) with K replaced by \tilde{K} . The particular choice (36) is too restrictive for quenches to integrable massive QFTs³⁷, and an important question is what kind of initial conditions can be described by an appropriate choice of \tilde{K} ^{37,38}. One might hope that an approximate realisation of such an initial state could be obtained in a quantum quench

$$\lambda \Big|_{t < 0} = \lambda_- \longrightarrow \lambda \Big|_{t > 0} = \lambda_+, \quad (40)$$

where λ_{\pm} are both large but finite, and their difference is small. This is clearly an important issue worthy of further investigation, but is beyond the scope of our present work.

In the following we will simply set aside the question of how our initial state can be prepared and impose it to be of the form (32), where $K^{ab}(\theta)$ is a solution of equations (33) and (34). We will furthermore use the freedom (35) to be able to consider $K^{ab}(\theta)$, and hence the quench, to be small in the sense of Refs. [5]. In Sections VID and VIE we will make the additional simplifying assumption that³⁹

$$K^{aa}(\theta) = 0, \quad (41)$$

which is for example the case for the initial condition $\Phi(t = 0, x) = 0$.

V. “REPRESENTATIVE STATE” APPROACH

In Ref. [16] a novel approach to calculating expectation values of local operators after quantum quenches to integrable models was proposed. It was subsequently applied to determine the infinite-time behaviour of certain local observables in an interaction quench from free to interacting bosons in the Lieb-Liniger model¹³ and very recently the XXZ chain⁴⁰, and is referred to as “quench-action” approach in these works. Using thermodynamic arguments it was argued in Ref. [16] that

$$\begin{aligned} \lim_{L \rightarrow \infty} \frac{L \langle \Psi_t | \mathcal{O}(x) | \Psi_t \rangle_L}{L \langle \Psi_t | \Psi_t \rangle_L} &= \lim_{L \rightarrow \infty} \frac{L \langle \Psi_0 | \mathcal{O}(t, x) | \Psi_0 \rangle_L}{L \langle \Psi_0 | \Psi_0 \rangle_L} \\ &= \lim_{L \rightarrow \infty} \frac{1}{2} \left[\frac{L \langle \Psi_0 | \mathcal{O}(t, x) | \Phi \rangle_L}{L \langle \Psi_0 | \Phi \rangle_L} + \frac{L \langle \Phi | \mathcal{O}(t, x) | \Psi_0 \rangle_L}{L \langle \Phi | \Psi_0 \rangle_L} \right], \end{aligned} \quad (42)$$

where $|\Psi_0\rangle_L$ is the initial state in a large, finite volume L , and $|\Phi\rangle_L$ is a *representative state* characterized by the requirements that

1. $|\Phi\rangle_L$ is an eigenstate with eigenvalue E_Φ of the Hamiltonian in a large, finite volume.
2. The expectation values of the macroscopically many (local) integrals of motion $I^{(n)}$ inherent to quantum integrable models⁴¹ are the same in the initial and the representative state

$$\lim_{L \rightarrow \infty} \frac{1}{L} \frac{L \langle \Phi | I^{(n)} | \Phi \rangle_L}{L \langle \Phi | \Phi \rangle_L} = \lim_{L \rightarrow \infty} \frac{1}{L} \frac{L \langle \Psi_0 | I^{(n)} | \Psi_0 \rangle_L}{L \langle \Psi_0 | \Psi_0 \rangle_L}, \quad n = 1, 2, \dots \quad (43)$$

In the representative state approach there are several main steps in determining the time evolution of local operators after quenches to integrable models:

- Construct a basis $\{|\theta_1, \dots, \theta_N\rangle_L^s | s = 1, \dots, 2^N\}$ of N -particle energy eigenstates in the finite volume.
- Construct finite-volume matrix elements of (semi-) local operators \mathcal{O} with respect to these eigenstates.
- Given an initial state $|\Psi_0\rangle$, determine a corresponding representative state $|\Phi\rangle_L$.
- For a given operator $\mathcal{O}(x)$, work out the time evolution using a Lehmann representation of (42) in terms of energy eigenstates, i.e. work out matrix elements of the form

$$\begin{aligned} {}_L\langle\Psi_0|\mathcal{O}(t,x)|\Phi\rangle_L &= \sum_{N\geq 0} \sum_{\{\theta_1,\dots,\theta_N\}_L^s} {}_L\langle\Psi_0|\theta_1,\dots,\theta_N\rangle_L^s {}_L\langle\theta_N,\dots,\theta_1|\mathcal{O}(t,x)|\Phi\rangle_L \\ &= \sum_{N\geq 0} \sum_{\{\theta_1,\dots,\theta_N\}_L^s} {}_L\langle\Psi_0|\theta_1,\dots,\theta_N\rangle_L^s {}_L\langle\theta_N,\dots,\theta_1|\mathcal{O}(0,0)|\Phi\rangle_L e^{it(E_\Phi - \sum_{j=1}^N \Delta \cosh \theta_j)}. \end{aligned} \quad (44)$$

We note that ultimately we are only interested in the limit $L \rightarrow \infty$ and use finite L only as a regularization. As a result we only take into account the dominant finite-size corrections.

A. Basis of energy eigenstates at large, finite volume

Our starting point for constructing energy eigenstates in the finite volume are the infinite-volume scattering states (28). The idea is to consider appropriate linear combinations of scattering states, and then to impose boundary conditions on these, see e.g. Ref. [42]. To this end, we introduce a *transfer matrix* acting on scattering states as follows

$$\mathcal{T}(\lambda|\{\theta_k\})|\theta_1, \dots, \theta_N\rangle_{a_1 \dots a_N} = \mathcal{T}(\lambda|\{\theta_k\})_{a_1 \dots a_N}^{b_1 \dots b_N} |\theta_1, \dots, \theta_N\rangle_{b_1 \dots b_N}, \quad (45)$$

where

$$\mathcal{T}(\lambda|\{\theta_k\})_{a_1 \dots a_N}^{b_1 \dots b_N} = \mathcal{S}_{c_N a_1}^{c_1 b_1}(\lambda - \theta_1) \dots \mathcal{S}_{c_{N-1} a_N}^{c_N b_N}(\lambda - \theta_N). \quad (46)$$

As the S-matrix is a solution of the Yang-Baxter equation, the transfer matrices form a commuting family

$$[\mathcal{T}(\lambda|\{\theta_k\}), \mathcal{T}(\mu|\{\theta_k\})] = 0. \quad (47)$$

Hence the transfer matrices can be diagonalized simultaneously

$$\mathcal{T}(\lambda|\{\theta_k\})|\theta_1, \dots, \theta_N\rangle^s = \Lambda^s(\lambda|\{\theta_k\})|\theta_1, \dots, \theta_N\rangle^s, \quad s = 1, \dots, 2^N. \quad (48)$$

We refer to the labels s as *polarisations*. Details of the construction of transfer-matrix eigenstates and explicit expressions for the eigenvalues $\Lambda^s(\lambda|\{\theta_k\})$ are given in Appendix B. By construction energy and momentum of the states (48) are still given by formulae (29). The basis transformation between scattering states and transfer matrix eigenstates can be cast in the form

$$|\theta_1, \dots, \theta_N\rangle^s = \sum_{a_1 \dots a_N} \Psi_{a_1 \dots a_N}^s(\{\theta_k\}) |\theta_1, \dots, \theta_N\rangle_{a_1 \dots a_N}, \quad (49)$$

$$|\theta_1, \dots, \theta_N\rangle_{a_1 \dots a_N} = \sum_s \Psi_{a_1 \dots a_N}^s(\{\theta_k\})^* |\theta_1, \dots, \theta_N\rangle^s, \quad (50)$$

where the amplitudes $\{\Psi_{a_1 \dots a_N}^s(\{\theta_k\})\}_{s=1, \dots, 2^N}$ satisfy

$$\mathcal{T}(\lambda|\{\theta_k\})_{a_1 \dots a_N}^{b_1 \dots b_N} \Psi_{a_1 \dots a_N}^s(\{\theta_k\}) = \Lambda^s(\lambda|\{\theta_k\}) \Psi_{b_1 \dots b_N}^s(\{\theta_k\}), \quad (51)$$

$$\sum_{a_1 \dots a_N} \Psi_{a_1 \dots a_N}^s(\{\theta_k\})^* \Psi_{a_1 \dots a_N}^r(\{\theta_k\}) = \delta_{rs}, \quad (52)$$

$$\sum_s \Psi_{a_1 \dots a_N}^s(\{\theta_k\})^* \Psi_{b_1 \dots b_N}^s(\{\theta_k\}) = \prod_{j=1}^N \delta_{a_j, b_j}. \quad (53)$$

The last two equations ensure that $\{|\theta_1, \dots, \theta_N\rangle^s\}_{s=1, \dots, 2^N}$ is an orthonormal and complete set. Since the topological charge operator Q commutes with the transfer matrix we can choose $\{\Psi_{a_1 \dots a_N}^s(\{\theta_k\})\}_{s=1, \dots, 2^N}$ such that Q is diagonal in the basis $\{|\theta_1, \dots, \theta_N\rangle^s\}_{s=1, \dots, 2^N}$

$$Q|\theta_1, \dots, \theta_N\rangle^s = Q(s)|\theta_1, \dots, \theta_N\rangle^s, \quad Q(s) \in \{N, N-2, \dots, -N\}. \quad (54)$$

We are now in a position to consider a large, finite volume L and impose periodic boundary conditions on the transfer matrix eigenstates

$$e^{i\ell \sinh \theta_i} |\theta_1, \dots, \theta_N\rangle_L^s = \mathcal{T}^{-1}(\theta_i|\{\theta_k\}) |\theta_1, \dots, \theta_N\rangle_L^s, \quad i = 1, \dots, N, \quad (55)$$

where we have introduced a dimensionless length $\ell = L\Delta/v$. Equations (55) lead to quantization conditions for the rapidities in the finite volume and are known as Bethe-Yang equations⁴²

$$e^{i\ell \sinh \theta_i} \Lambda^s(\theta_i|\{\theta_k\}) = 1, \quad i = 1, \dots, N, \quad s = 1, \dots, 2^N. \quad (56)$$

For practical purposes the logarithmic version of (56) is more convenient

$$2\pi I_i^s = \ell \sinh \theta_i - i \log \Lambda^s(\theta_i|\{\theta_k\}) \equiv Q_i^s(\theta_1, \dots, \theta_N), \quad i = 1, \dots, N, \quad s = 1, \dots, 2^N. \quad (57)$$

Here I_i^s are integer or half-odd integer numbers that uniquely specify a given solution and concomitantly the corresponding transfer matrix eigenstate. By virtue of these facts one may use the integers (rather than the rapidities) as labels for constructing a basis of eigenstates⁴³. The latter is given by the states

$$| \{I_1, \dots, I_N\} \rangle_L^s, \quad I_1 < \dots < I_N, \quad (58)$$

where we impose normalisation conditions

$${}_L^r \langle \{J_M, \dots, J_1\} | \{I_1, \dots, I_N\} \rangle_L^s = \delta_{MN} \delta_{rs} \delta_{J_1 I_1} \dots \delta_{J_N I_N}. \quad (59)$$

In a given sector specified by N and s the Jacobian matrix of the (invertible) mapping between rapidities $\{\theta_k\}_{k=1, \dots, N}$ and integers $\{I_k\}_{k=1, \dots, N}$ is given by

$$\mathcal{J}_N^s(\theta_1, \dots, \theta_N)_{ij} = \partial_{\theta_j} Q_i^s(\theta_1, \dots, \theta_N). \quad (60)$$

The N -particle density of states with polarisation s is defined as the Jacobian

$$\rho_N^s(\theta_1, \dots, \theta_N) = \det \mathcal{J}_N^s(\theta_1, \dots, \theta_N). \quad (61)$$

B. 2-folded sine-Gordon model

When quantising the theory in the finite volume, care has to be taken to account for the absence of spontaneous symmetry breaking of the discrete symmetry

$$\Phi(x) \longrightarrow \Phi(x) + \frac{2\pi n}{\beta}, \quad n \in \mathbb{Z} \quad (62)$$

of the sine-Gordon Hamiltonian. In the applications of the sine-Gordon model we have in mind, the Bose field is compactified on a ring with radius

$$R = \frac{4\pi}{\beta}. \quad (63)$$

This implies that on a classical level there are two vacuum states $|0\rangle$ and $|1\rangle$, with corresponding field expectation values

$$\langle n | \Phi(x, t) | n \rangle = \frac{2\pi}{\beta} n, \quad n = 0, 1. \quad (64)$$

This theory is known as the 2-folded sine-Gordon model $\text{SG}(\beta, 2)$, and has been analysed in some detail in Ref. [44]. We now review some relevant results. On the quantum level, the two ground states are mapped one into one other by the transformation

$$T : \Phi(x) \longrightarrow \Phi(x) - \frac{2\pi}{\beta}, \quad (65)$$

which is a symmetry of the Hamiltonian. The linear combinations of $|0\rangle$ and $|1\rangle$ that are eigenvectors of T are given by

$$|0\rangle_{\text{R,NS}} = \frac{|0\rangle \pm |1\rangle}{\sqrt{2}}. \quad (66)$$

In the infinite volume, one can construct scattering states over each of the two ground states. They are denoted by

$$|\theta_1, \dots, \theta_N\rangle_{a_1 \dots a_N}^n \quad (67)$$

where $a_i = \pm 1$ are topological charge quantum numbers and $n = 0, 1$ label the two ground states. In order to define the $\text{SG}(\beta, 2)$ model in a finite volume, one considers eigenstates of the “shift”-operator T defined above

$$|\theta_1, \dots, \theta_N\rangle_{a_1 \dots a_N}^{\text{R}} = \frac{1}{\sqrt{2}} \left\{ |\theta_1, \dots, \theta_N\rangle_{a_1 \dots a_N}^0 + |\theta_1, \dots, \theta_N\rangle_{a_1 \dots a_N}^1 \right\}, \quad (68)$$

$$|\theta_1, \dots, \theta_N\rangle_{a_1 \dots a_N}^{\text{NS}} = \frac{1}{\sqrt{2}} \left\{ |\theta_1, \dots, \theta_N\rangle_{a_1 \dots a_N}^0 - |\theta_1, \dots, \theta_N\rangle_{a_1 \dots a_N}^1 \right\}. \quad (69)$$

In a finite volume, periodic boundary conditions select only states for which the set $\{a_i\}$ is subject to the constraint

$$\sum_{i=1}^N a_i \equiv 0 \pmod{2}. \quad (70)$$

Condition (70) is equivalent to requiring that the number N of particles be even. States involving an odd number of (anti) solitons are incompatible with the boundary conditions and are not part of the Hilbert space. Nevertheless all *local* properties of $\text{SG}(\beta, 2)$ coincide with the corresponding quantities in the sine-Gordon model. In order to impose boundary conditions we again go over to transfer matrix eigenstates, which we denote by

$$|\theta_1, \dots, \theta_N\rangle_{\mathbf{a}}^s, \quad \mathbf{a} = \text{R, NS}. \quad (71)$$

Periodic boundary conditions imply that

$$e^{i\ell \sinh \theta_i} |\theta_1, \dots, \theta_N\rangle_{\mathbf{a}}^s = \sigma_{\mathbf{a}} \mathcal{T}^{-1}(\theta_i | \{\theta_k\}) |\theta_1, \dots, \theta_N\rangle_{\mathbf{a}}^s, \quad i = 1, \dots, N, \quad (72)$$

where $\sigma_{\text{R}} = 1 = -\sigma_{\text{NS}}$. Eqs. (72) lead to finite-volume quantisation conditions of the form

$$e^{i\ell \sinh \theta_i} \Lambda^s(\theta_i | \{\theta_k\}) = \sigma_{\mathbf{a}}, \quad i = 1, \dots, N, \quad s = 1, \dots, 2^N, \quad \mathbf{a} = \text{R, NS}. \quad (73)$$

Taking the logarithm we obtain

$$Q_i^s(\theta_1, \dots, \theta_N) = \ell \sinh \theta_i - i \log \Lambda^s(\theta_i | \{\theta_k\}) = 2\pi (I_i^s + \kappa_{\mathbf{a}}), \quad \kappa_{\mathbf{a}} \equiv \frac{1 - \sigma_{\mathbf{a}}}{4}. \quad (74)$$

Using again the (half-odd) integers $\{I_k\}$ to label the states we denote our basis of transfer matrix eigenstates in a large, finite volume L by

$$\{|\{I_1, \dots, I_N\}\rangle_{\mathbf{a}}^s, \quad N \text{ even}, \quad s = 1, \dots, 2^N, \quad \mathbf{a} = \text{R, NS}\}. \quad (75)$$

Here distinct sets of I_j give rise to different basis states, and a natural choice would therefore be $I_1 < I_2 < \dots < I_N$. However, in the following we will be interested in the case where our solutions consist of pairs, i.e. $\{-I_1, I_1, -I_2, I_2, \dots\}$. Having this situation in mind, we choose the following odd-looking convention for labelling our basis states

$$I_{N-1} < I_{N-3} < \dots < I_3 < I_1 < I_2 < I_4 < \dots < I_{N-2} < I_N. \quad (76)$$

Before concluding this subsection we note that the operator $\mathcal{O}(t, x) = \exp(i\beta\Phi(t, x)/2)$ has non-vanishing matrix elements only between different sectors. This follows because the operator is odd under the symmetry T

$$T\mathcal{O}(t, x)T^\dagger = -\mathcal{O}(t, x), \quad (77)$$

while the states are either even or odd under T .

C. Matrix elements of local operators in the finite volume

A general method for determining matrix elements of local operators in a large, finite volume was developed in Refs. [42] and [43]. The leading corrections to the form factors in the infinite-volume limit arises from the quantisation of the rapidities. Given infinite-volume form factors $f_{b_1 \dots b_N}^{\mathcal{O}^{a_M \dots a_1}}(\theta_M, \dots, \theta_1 | \tilde{\theta}_1, \dots, \tilde{\theta}_N)$ of an operator $\mathcal{O}(x)$, one first forms appropriate linear combinations

$$f^{\mathcal{O}}(\theta_M, \dots, \theta_1 | \tilde{\theta}_1, \dots, \tilde{\theta}_N)_s^s = \sum_{\substack{a_1 \dots a_M \\ b_1 \dots b_N}} \Psi_{b_1 \dots b_N}^{\tilde{s}}(\{\tilde{\theta}_k\}) \Psi_{a_1 \dots a_M}^s(\{\theta_k\})^* f_{b_1 \dots b_N}^{\mathcal{O}^{a_M \dots a_1}}(\theta_M, \dots, \theta_1 | \tilde{\theta}_1, \dots, \tilde{\theta}_N), \quad (78)$$

where the amplitudes $\Psi_{a_1 \dots a_M}^s(\{\theta_k\})$ are defined in (49). Matrix elements in the basis (75) for operators \mathcal{O} odd under the transformation T are then given by

$${}_R^s \langle \{I_M, \dots, I_1\} | \mathcal{O}(0, 0) | \{\tilde{I}_1, \dots, \tilde{I}_N\} \rangle_{NS}^{\tilde{s}} = \frac{f^{\mathcal{O}}(\theta_M, \dots, \theta_1 | \tilde{\theta}_1, \dots, \tilde{\theta}_N)_s^s}{\sqrt{\rho_M^s(\theta_1, \dots, \theta_M) \rho_N^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N)}} + \mathcal{O}(e^{-\mu' L}). \quad (79)$$

Here N and M are both even in our case.

D. Finite-volume regularisation of integrable boundary states

The next step is to obtain a realisation of our initial state $|\Psi\rangle_0$ in a large, finite volume L . Given that in the thermodynamic limit we must reproduce the spontaneous breaking of the symmetry (62), the appropriate linear combination is given by (cf. Eq. (66))

$$|\Psi_0\rangle_L = \frac{1}{\sqrt{2}} [|\Psi_0\rangle_R + |\Psi_0\rangle_{NS}]. \quad (80)$$

We now wish to express $|\Psi_0\rangle_a$ in terms of transfer-matrix eigenstates. This can be done by generalising the results of Ref. [45], which considered finite-volume realisations of integrable boundary states for diagonal scattering theories, for which both the scattering matrix $S_{cd}^{ab}(\theta)$ and the $K^{ab}(\theta)$ are scalars. This leads to an expression of the form

$$|\Psi_0\rangle_a = \sum_{N=0}^{\infty} \sum_{s=1}^{2^{2N}} \sum_{0 < I_1 < \dots < I_N} \mathcal{N}_{2N}^s(\theta_1, \dots, \theta_N) \mathcal{K}_{2N}^s(\theta_1, \dots, \theta_N) | \{-I_1, I_1, \dots, -I_N, I_N\} \rangle_a^s, \quad a = R, NS. \quad (81)$$

We emphasize that the rapidities appearing in this formula are the parity symmetric solutions $\{-\theta_1, \theta_1, \dots, -\theta_N, \theta_N\}$ of the Bethe-Yang equations for every fixed N , s and a , with $0 < \theta_1 < \dots < \theta_N$. The functions \mathcal{N}_{2N}^s and \mathcal{K}_{2N}^s in (81) are defined as

$$\mathcal{K}_{2N}^s(\theta_1, \dots, \theta_N) \equiv K^{a_1 b_1}(\theta_1) \dots K^{a_N b_N}(\theta_N) \Psi_{a_1 b_1 \dots a_N b_N}^s(\{\theta_k\})^*, \quad (82)$$

$$\mathcal{N}_{2N}^s(\theta_1, \dots, \theta_N) \equiv \frac{\sqrt{\rho_{2N}^s(-\theta_1, \theta_1, \dots, -\theta_N, \theta_N)}}{\bar{\rho}_N^s(\theta_1, \dots, \theta_N)}, \quad (83)$$

while $\bar{\rho}_N^s(\theta_1, \dots, \theta_N)$ is the constrained N -particle density of states with polarisation s

$$\bar{\rho}_N^s(\theta_1, \dots, \theta_N) = \det \bar{\mathcal{J}}_N^s(\theta_1, \dots, \theta_N), \quad \bar{\mathcal{J}}_N^s(\theta_1, \dots, \theta_N)_{ij} = \partial_{\theta_j} \bar{Q}_i^s(\theta_1, \dots, \theta_N). \quad (84)$$

Here $\bar{Q}_i^s(\theta_1, \dots, \theta_N)$ is the function Q_i^s in the sector with $2N$ -particles and polarisation s evaluated for a symmetric rapidity distribution (so it refers to a particular subset of solutions of the Bethe-Yang equations for $2N$ particles)

$$\bar{Q}_i^s(\theta_1, \dots, \theta_N) = Q_i^s(-\theta_1, \theta_1, \dots, -\theta_N, \theta_N), \quad i = 1, \dots, N. \quad (85)$$

Expression (81) is obtained, following Ref. [45], by imposing that the expectation value of an arbitrary local operator $\mathcal{O}(x)$ (even under T) in the state $|\Psi_0\rangle_a$ must reproduce the infinite-volume result up to exponentially small corrections in system size. Inserting resolutions of the identity in terms of the basis $\{|I_1, \dots, I_N\rangle_a^s\}$, and then using the analogue of relations (79) to compute the matrix elements, we obtain (81). A complication that arises compared to the case of

diagonal scattering is that not every sector s allows parity symmetric rapidity distributions of the kind used in (81). Indeed, such solutions exist only if the transfer matrix eigenvalues fulfil the relation

$$\Lambda^s(-\theta_i|\{-\theta_k, \theta_k\}) = \Lambda^s(\theta_i|\{-\theta_k, \theta_k\})^{-1}. \quad (86)$$

How about sectors where (86) does not hold? It turns out that they do not contribute to the expansion of the boundary state. To see this, let us go back to the infinite volume, where we have [cf. (50)]

$$|\Psi_0\rangle = \sum_{N=0}^{\infty} \sum_{s=1}^{2^{2N}} \int \frac{d\theta_1}{2\pi} \int_{\theta_1}^{\infty} \frac{d\theta_2}{2\pi} \cdots \int_{\theta_{N-1}}^{\infty} \frac{d\theta_N}{2\pi} \mathcal{K}_{2N}^s(\theta_1, \dots, \theta_N) |-\theta_1, \theta_1, \dots, -\theta_N, \theta_N\rangle^s. \quad (87)$$

In Appendix C we demonstrate that

$$\mathcal{K}_{2N}^s(\theta_1, \dots, \theta_N) \Lambda^s(-\theta_i|\{-\theta_k, \theta_k\}) = \mathcal{K}_{2N}^s(\theta_1, \dots, \theta_N) \Lambda^s(\theta_i|\{-\theta_k, \theta_k\})^{-1}. \quad (88)$$

This implies that either (86) holds, in which case parity-symmetric solutions exist, or the coefficients $\mathcal{K}_{2N}^s(\theta_1, \dots, \theta_N)$ must vanish, in which case the corresponding sector s does not contribute to the expansion of the boundary state.

E. Determination of the representative eigenstate

The final step required to set up our calculation of expectation values of local operators is to obtain an expression for the representative state. In Ref.[16] a method for constructing the representative state for a given $|\Psi_0\rangle$ via a generalized thermodynamic Bethe Ansatz⁴⁶ was presented. Here we follow a different route. By definition $|\Phi\rangle_L$ is a finite-volume basis state (75), which fulfils the requirements that

$$\lim_{L \rightarrow \infty} \frac{1}{L} \frac{{}_L\langle \Phi | I_L^{(n)} | \Phi \rangle_L}{{}_L\langle \Phi | \Phi \rangle_L} = \lim_{L \rightarrow \infty} \frac{1}{L} \frac{{}_L\langle \Psi_0 | I_L^{(n)} | \Psi_0 \rangle_L}{{}_L\langle \Psi_0 | \Psi_0 \rangle_L}, \quad n = 1, 2, \dots \quad (89)$$

Here $\{I_L^{(n)}\}$ are suitable finite-volume regularisations of the conservation laws $\{I^{(n)}\}$. As already pointed out in Ref. [16], one may use (89) to determine the root density $\rho_\Phi(\theta)$ specifying the representative state. Here the root density corresponding to a solution $\{\theta_1, \dots, \theta_N\}$ of the Bethe-Yang equations such that $\theta_{j+1} - \theta_j = \mathcal{O}(L^{-1})$ for large L is defined as

$$\rho(\theta_j) = \lim_{L \rightarrow \infty} \frac{1}{L(\theta_{j+1} - \theta_j)}. \quad (90)$$

Given the density $\rho_\Phi(\theta)$, a particular representative microstate can be constructed. It takes the form

$$|\Phi\rangle_L = |\{-\tilde{I}_1, \tilde{I}_1, \dots, -\tilde{I}_N, \tilde{I}_N\}\rangle_{\text{NS}}^{\tilde{s}} \equiv |\Phi_{\tilde{s}}\rangle_{\text{NS}}, \quad (91)$$

where

- the number of rapidities N is given by $2N = \lceil \delta L \rceil$, where $\delta = \int_{\mathbb{R}} d\theta \rho_\Phi(\theta)$ is the total density of particles;
- we have chosen the state to occur in the NS sector;
- the set of integers $\{\tilde{I}_k\}$ is such that the corresponding root density is equal to $\rho_\Phi(\theta)$;
- the sector \tilde{s} is chosen such that (86) holds, i.e. symmetric solutions of the Bethe-Yang solutions exist. Furthermore we require the topological charge to fulfil $Q(\tilde{s}) = 2m = \lceil qL \rceil + k$ where $k \in \mathbb{Z}$ fixed and q is the density of the expectation value of the topological charge in the initial state ($|q| \leq \delta$).

It turns out that in the regime we are working in (low densities) we do not require an explicit expression of q in terms of $K^{ab}(\theta)$ (our “initial” data), because at late times the leading contribution to the expectation value (42) does not depend on the value of q .

1. Calculation of the root density

We start by rewriting the conditions (89) in terms of the root density $\rho_\Phi(\theta)$

$$\lim_{L \rightarrow \infty} \frac{1}{L} \frac{L \langle \Psi_0 | I_L^{(n)} | \Psi_0 \rangle_L}{L \langle \Psi_0 | \Psi_0 \rangle_L} = \int_{-\infty}^{\infty} d\theta \rho_\Phi(\theta) i^{(n)}(\theta), \quad n = 1, 2, \dots, \quad (92)$$

where the functions $i^{(n)}(\theta)$ parameterize the eigenvalues of the conservation laws $I_L^{(n)}$

$$I_L^{(n)} |\{I_1, \dots, I_N\}\rangle_{\mathbf{a}}^s = \left\{ \sum_{i=1}^N i^{(n)}(\theta_i) \right\} |\{I_1, \dots, I_N\}\rangle_{\mathbf{a}}^s, \quad (93)$$

Here $\{\theta_1, \dots, \theta_N\}$ is the solution of the Bethe-Yang equations in sector s corresponding to the integers I_1, \dots, I_N . A set of local conservation laws for the sine-Gordon model is in principle known⁴⁷. However, obtaining the root density from an explicit calculation of the expectation values of these conserved charges is a very challenging problem (assuming that the local conservation laws are sufficient). Here we proceed in a different way. The idea is to use (92) in reverse: if the functions $i^{(n)}$ form a complete set, we may determine $\rho_\Phi(\theta)$ from a known set of expectation values of conservation laws $I^{(n)}$. Assuming this to be the case, the requirements (92) are equivalent to

$$\lim_{L \rightarrow \infty} \frac{1}{L} \frac{L \langle \Psi_0 | \hat{N}_\zeta | \Psi_0 \rangle_L}{L \langle \Psi_0 | \Psi_0 \rangle_L} = \int_{-\infty}^{\infty} d\theta \rho_\Phi(\theta) \zeta(\theta), \quad (94)$$

where

$$\hat{N}_\zeta |\{I_1, \dots, I_N\}\rangle_{\mathbf{a}}^s = \left\{ \sum_{i=1}^N \zeta(\theta_i) \right\} |\{I_1, \dots, I_N\}\rangle_{\mathbf{a}}^s, \quad (95)$$

and $\zeta(x)$ is an *arbitrary* function in the space spanned by the $i^{(n)}(\theta)$. Our procedure essentially amounts to starting with the conserved “mode-occupation numbers” in the thermodynamic limit $I(\theta) = \sum_a Z_a^\dagger(\theta) Z_a(\theta)$. Given that the $I(\theta)$ are conserved, it follows that

$$\hat{N}_\zeta = \int d\theta \zeta(\theta) \sum_a Z_a^\dagger(\theta) Z_a(\theta) \quad (96)$$

are conserved as well. The idea is to find an appropriate finite volume regularization of these operators, which together with the arbitrariness of $\zeta(\theta)$ can be used to determine ρ_Φ from conditions (94).

It is convenient to express \hat{N}_ζ as

$$\hat{N}_\zeta \equiv \sum_{I \in \mathbb{Z}} \hat{n}_\zeta(I), \quad (97)$$

where $\hat{n}_\zeta(I)$ acts on the basis (75) as follows

$$\hat{n}_\zeta(I) |\{I_1, \dots, I_N\}\rangle_{\mathbf{a}}^s = \left\{ \sum_{i=1}^N \delta_{I_i, I + \kappa_{\mathbf{a}}} \zeta(\theta_i) \right\} |\{I_1, \dots, I_N\}\rangle_{\mathbf{a}}^s. \quad (98)$$

Here $\kappa_{\mathbf{a}}$ have been defined in (74). Using the form (80) for the initial state in the finite volume and fixing $I > 0$ we obtain

$$L \langle \Psi_0 | \hat{n}_\zeta(I) | \Psi_0 \rangle_L = \frac{1}{2} \sum_{\mathbf{a}} \sum_{N'=1}^{\infty} \sum_{s=1}^{2^{N'}} \sum_{m=1}^{N'} \sum_{\{I_1, \dots, I_{N'}\}_I^m} \mathcal{N}_{2N'}^s(\theta_1, \dots, \theta_{N'})^2 |\mathcal{K}_{2N'}^s(\theta_1, \dots, \theta_{N'})|^2 \zeta(\theta_m), \quad (99)$$

where $\{I_1, \dots, I_{N'}\}_I^m$ denotes the set $\{I_1, \dots, I_{N'}\}$ with I_m removed and the integers are ordered as

$$0 < \underbrace{I_1 < \dots < I + \kappa_{\mathbf{a}}}_{m} < \dots < I_{N'}. \quad (100)$$

The case $I < 0$ is dealt with analogously. Our strategy is now to convert the sum

$$\frac{1}{L} \sum_{I \in \mathbb{Z}} \frac{{}_L \langle \Psi_0 | \hat{n}_\zeta(I) | \Psi_0 \rangle_L}{{}_L \langle \Psi_0 | \Psi_0 \rangle_L}, \quad (101)$$

into an integral in the thermodynamic limit, and then to use the arbitrariness of the function $\zeta(\theta)$ to determine $\rho_\Phi(\theta)$. In order for this to be possible we have to compute

$$\frac{{}_L \langle \Psi_0 | \hat{n}_\zeta(I) | \Psi_0 \rangle_L}{{}_L \langle \Psi_0 | \Psi_0 \rangle_L} \quad (102)$$

and show that is a function of I/L . The problem of calculating (102) exactly still presents a formidable task. We therefore restrict our attention to the case of “small” quenches in the sense of Refs. [5],[17], namely to cases where the densities of excitations of the post-quench Hamiltonian in the initial state are small. Then we may use $|K^{ab}(\theta)|$ as formal expansion parameters, and determine (102) by means of a linked-cluster expansion first introduced for the finite-temperature case in Refs. [49]. We start by expanding the denominator in (102) as

$$\begin{aligned} {}_L \langle \Psi_0 | \Psi_0 \rangle_L &= 1 + \sum_{n \geq 1} \Upsilon_{2N}, \\ \Upsilon_{2N} &\equiv \frac{1}{2} \sum_{\mathbf{a}} \sum_{s=1}^{2^{2N}} \sum_{0 < I_1 < \dots < I_N} \mathcal{N}_{2N}^s(\theta_1, \dots, \theta_N)^2 |\mathcal{K}_{2N}^s(\theta_1, \dots, \theta_N)|^2. \end{aligned} \quad (103)$$

The rapidities appearing in the expression for Υ_{2N} are solutions of the Bethe-Yang equations in the sector determined by N , s and \mathbf{a} . Denoting the contribution at order $2M$ in $|K^{ab}(\theta)|$ in (99) by $\mathcal{C}_{\zeta,I}^{2M}$, i.e.

$${}_L \langle \Psi_0 | \hat{n}_\zeta(I) | \Psi_0 \rangle_L = \sum_{M \geq 1} \mathcal{C}_{\zeta,I}^{2M}, \quad (104)$$

we define *linked clusters* $\mathcal{D}_{\zeta,I}^{2M}$ recursively by

$$\mathcal{C}_{\zeta,I}^{2M} = \mathcal{D}_{\zeta,I}^{2M} + \sum_{N=1}^{M-1} \Upsilon_{2N} \mathcal{D}_{\zeta,I}^{2(M-N)}, \quad M = 1, 2, \dots \quad (105)$$

In terms of the linked clusters our quantity of interest reads

$$\frac{{}_L \langle \Psi_0 | \hat{n}_\zeta(I) | \Psi_0 \rangle_L}{{}_L \langle \Psi_0 | \Psi_0 \rangle_L} = \sum_{N \geq 1} \mathcal{D}_{\zeta,I}^{2N}. \quad (106)$$

Under the assumption that for small quenches (106) has a well-defined low-density expansion, the leading contribution is simply given by the first term in the series

$$\mathcal{D}_{\zeta,I}^2 = \frac{1}{2} \sum_{\mathbf{a}} \sum_{s=1}^{2^2} \mathcal{N}_2^s(\theta_{\mathbf{a}}^s(I)) |\mathcal{K}_2^s(\theta_{\mathbf{a}}^s(I))|^2 \zeta(\theta_{\mathbf{a}}^s(I)). \quad (107)$$

Here $\theta_{\mathbf{a}}^s(I)$ is the solution of the Bethe-Yang equations

$$\bar{Q}^s(\theta) = 2\pi(I + \kappa_{\mathbf{a}}). \quad (108)$$

Up to finite-size corrections, the Bethe-Yang equations can be easily solved

$$\theta_{\mathbf{a}}^s(I) = \theta(I) + \mathcal{O}(1/\ell), \quad \theta(I) = \operatorname{arcsinh}(2\pi I/\ell). \quad (109)$$

Noting that $\mathcal{N}_2^s(\theta_{\mathbf{a}}^s(I)) = 1 + \mathcal{O}(1/\ell) \forall s, \mathbf{a}$, we have

$$\mathcal{D}_{\zeta,I}^2 = \zeta(\theta(I)) G(\theta(I)) + \mathcal{O}(1/\ell), \quad (110)$$

where we introduced

$$G(\theta) \equiv \sum_{a,b=\pm} |K^{ab}(\theta)|^2. \quad (111)$$

The result for the expectation value (101) is then

$$\frac{L \langle \Psi_0 | \hat{n}_\zeta(I) | \Psi_0 \rangle_L}{L \langle \Psi_0 | \Psi_0 \rangle_L} = \zeta(\theta(I)) G(\theta(I)) + \mathcal{O}(K^4). \quad (112)$$

The sum of this expression over all integers can be expressed as an integral using contour integration methods

$$\lim_{L \rightarrow \infty} \frac{1}{L} \frac{L \langle \Psi_0 | \hat{N}_\zeta | \Psi_0 \rangle_L}{L \langle \Psi_0 | \Psi_0 \rangle_L} \simeq \lim_{L \rightarrow \infty} \sum_I \oint_{\mathcal{C}_{\theta(I)}} \frac{d\eta}{2\pi} \frac{\Delta}{v} \cosh \eta \frac{G(\eta) \zeta(\eta)}{e^{i\ell \sinh \eta} - 1}, \quad (113)$$

where the contour $\mathcal{C}_{\theta(I)}$ encircles $\theta(I)$. Since the integrand is analytic except for the residues at $\{\theta(I)\}$, we may join the paths $\mathcal{C}_{\theta(I)}$ and obtain a single contour \mathcal{C} encircling the real axis

$$\lim_{L \rightarrow \infty} \frac{1}{L} \frac{L \langle \Psi_0 | \hat{N}_\zeta | \Psi_0 \rangle_L}{L \langle \Psi_0 | \Psi_0 \rangle_L} \simeq \lim_{L \rightarrow \infty} \oint_{\mathcal{C}} \frac{d\theta}{2\pi} \frac{\Delta}{v} \cosh \theta \frac{G(\theta) \zeta(\theta)}{e^{i\ell \sinh \theta} - 1} = \frac{\Delta}{v} \int_{-\infty}^{+\infty} \frac{d\theta}{2\pi} G(\theta) \zeta(\theta) \cosh \theta. \quad (114)$$

In the last step we have used that the contribution from the path below the real axis is exponentially suppressed in ℓ , and will not contribute in the thermodynamic limit. Finally, equating (114) and (94) and then using the arbitrariness of $\zeta(\theta)$, we are able to find the root density $\rho_\Phi(\theta)$

$$\rho_\Phi(\theta) = \frac{\Delta}{2\pi v} G(\theta) \cosh \theta + \mathcal{O}(K^4). \quad (115)$$

F. Time evolution of the expectation value

With an expression for the representative eigenstate in hand, we are now in a position to write the explicit Lehmann representation for the expectation value (42) and work out its time evolution. We focus on the first term on the right hand side of Eq. (42), as the other can be obtained by taking complex conjugation combined with the change $\beta \rightarrow -\beta$. The Lehmann representation is given by

$$\begin{aligned} \frac{{}_R \langle \Psi_0 | e^{i\beta \Phi(t,x)/2} | \Phi_{\tilde{s}} \rangle_{\text{NS}}}{{}_\text{NS} \langle \Psi_0 | \Phi_{\tilde{s}} \rangle_{\text{NS}}} &= \sum_{M \geq 1} \sum_{s=1}^{2^{2M}} \sum_{0 < I_1 < \dots < I_M} \frac{\mathcal{N}_{2M}^s(\theta_1, \dots, \theta_M) \mathcal{K}_{2M}^s(\theta_1, \dots, \theta_M)}{\mathcal{N}_{2N}^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N) \mathcal{K}_{2N}^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N)} \exp \left[2it\Delta \left(\sum_{i=1}^M \cosh \theta_i - \sum_{i=1}^N \cosh \tilde{\theta}_i \right) \right] \\ &\times {}^s \langle \{I_M, -I_M, \dots, I_1, -I_1\} | e^{i\beta \Phi(0,0)/2} | \{-\tilde{I}_1, \tilde{I}_1, \dots, -\tilde{I}_N, \tilde{I}_N\} \rangle_{\text{NS}}^{\tilde{s}}, \end{aligned} \quad (116)$$

where $\{I_k\}$ and $\{\tilde{I}_k\}$ are respectively integers and half-odd integers, while the rapidities $\{\theta_k\}_{k=1, \dots, M}$ and $\{\tilde{\theta}_k\}_{k=1, \dots, N}$ are the corresponding solutions of the Bethe-Yang equations

$$\bar{Q}_i^s(\theta_1, \dots, \theta_M) = 2\pi I_i, \quad i = 1, \dots, M, \quad (117)$$

$$\bar{Q}_j^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N) = 2\pi \tilde{I}_j, \quad j = 1, \dots, N. \quad (118)$$

As discussed before, both sectors $(2M, s, \text{R})$ and $(2N, \tilde{s}, \text{NS})$ permit parity symmetric solutions. The matrix elements are related to infinite-volume form factors by (79), which in the case of interest reads

$${}^s \langle \{I_M, \dots, -I_1\} | e^{i\beta \Phi(0,0)/2} | \{-\tilde{I}_1, \dots, \tilde{I}_N\} \rangle_{\text{NS}}^{\tilde{s}} = \frac{f^{\beta/2}(\theta_M, \dots, -\theta_1 | -\tilde{\theta}_1, \dots, \tilde{\theta}_N)_{\tilde{s}}^s}{\sqrt{\rho_{2M}^s(-\theta_1, \dots, \theta_M) \rho_{2N}^{\tilde{s}}(-\tilde{\theta}_1, \dots, \tilde{\theta}_N)}} + \mathcal{O}(e^{-\mu' L}). \quad (119)$$

Using the crossing symmetry in the infinite-volume form factors gives an expression of the form

$$\begin{aligned} f^{\beta/2}(\theta_M, \dots, -\theta_1 | -\tilde{\theta}_1, \dots, \tilde{\theta}_N)_{\tilde{s}}^s &= \sum_{\substack{a_1 \dots a_{2M} \\ b_1 \dots b_{2N}}} \Psi_{b_1 \dots b_{2N}}^{\tilde{s}}(\{\tilde{\theta}_k\}) \Psi_{a_1 \dots a_{2M}}^s(\{\theta_k\})^* \\ &\times f_{\bar{a}_{2M} \dots \bar{a}_1, b_1 \dots b_{2N}}^{\beta/2}(\theta_M + i\pi, \dots, -\theta_1 + i\pi, -\tilde{\theta}_1, \dots, \tilde{\theta}_N) \\ &= f^{\beta/2}(\theta_M + i\pi, \dots, -\theta_1 + i\pi, -\tilde{\theta}_1, \dots, \tilde{\theta}_N)_{s, \tilde{s}}. \end{aligned} \quad (120)$$

In the following we will take into account only the contribution to (116) that arises from states with¹⁶ $N = M$, as this represents the leading one. It is shown in Appendix E that the contribution from states with $N = M$ dominates over those from states with $M > N$ at late times. The same conclusion holds also in the case of contributions from states with $M < N$ and can be proven analogously.

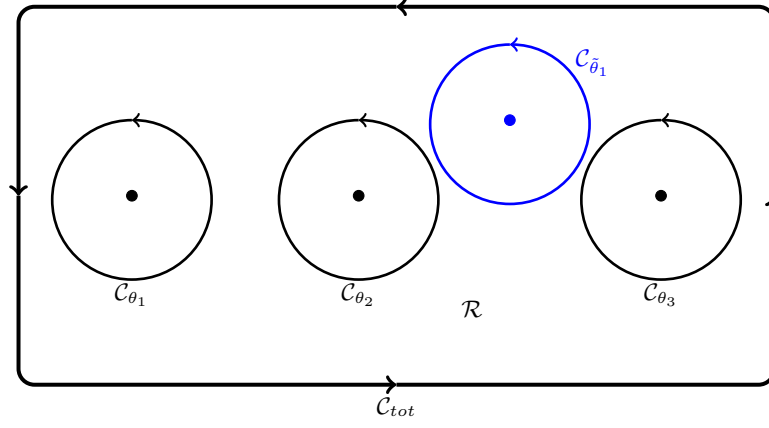


FIG. 1. Example of the contours in the one-variable case.

1. Contributions from states with $M = N$

Retaining only states with $M = N$ we have

$$\frac{\text{R}\langle\Psi_0|e^{i\beta\Phi(t,x)/2}|\Phi_{\tilde{s}}\rangle_{\text{NS}}}{\text{NS}\langle\Psi_0|\Phi_{\tilde{s}}\rangle_{\text{NS}}} \approx \sum_{s=1}^{2^{2N}} \sum_{0 < I_1 < \dots < I_N} \frac{\bar{\rho}_N^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N)}{\bar{\rho}_N^{\tilde{s}}(\theta_1, \dots, \theta_N) \rho_{2N}^{\tilde{s}}(-\tilde{\theta}_1, \dots, \tilde{\theta}_N)} \frac{\mathcal{K}_{2N}^s(\theta_1, \dots, \theta_N)}{\mathcal{K}_{2N}^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N)} \times e^{2i\Delta t \sum_{i=1}^N [\cosh \theta_i - \cosh \tilde{\theta}_i]} f^{\beta/2}(\theta_N + i\pi, \dots, \tilde{\theta}_N)_{s, \tilde{s}}. \quad (121)$$

Following the method introduced by Pozsgay and Takacs in [50], we rewrite each term in (121) as an integral on an appropriate multi-contour \mathcal{C} in \mathbb{C}^N by means of the multidimensional residue calculus

$$\frac{\text{R}\langle\Psi_0|e^{i\beta\Phi(t,x)/2}|\Phi_{\tilde{s}}\rangle_{\text{NS}}}{\text{NS}\langle\Psi_0|\Phi_{\tilde{s}}\rangle_{\text{NS}}} \approx \sum_{s=1}^{2^{2N}} \sum_{0 < I_1 < \dots < I_N} \oint_{\mathcal{C}_{\{\theta_\ell\}}} \prod_{k=1}^N \frac{d\eta_k}{2\pi} \frac{\bar{\rho}_N^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N)}{\rho_{2N}^{\tilde{s}}(-\tilde{\theta}_1, \dots, \tilde{\theta}_N)} \frac{\mathcal{K}_{2N}^s(\eta_1, \dots, \eta_N)}{\mathcal{K}_{2N}^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N)} \times e^{2i\Delta t \sum_{i=1}^N [\cosh \eta_i - \cosh \tilde{\theta}_i]} \frac{f^{\beta/2}(\eta_N + i\pi, \dots, \tilde{\theta}_N)_{s, \tilde{s}}}{\prod_{k=1}^N \{e^{iQ_k^s(\eta_1, \dots, \eta_N)} - 1\}}. \quad (122)$$

Here $\mathcal{C}_{\{\theta_\ell\}} = \mathcal{C}_{\theta_1} \otimes \dots \otimes \mathcal{C}_{\theta_N}$, where the contour \mathcal{C}_{θ_i} encircles the rapidity θ_i , i.e. the i -th rapidity in the solution $\{\theta_k\}$ of the Bethe-Yang equations in the sector $(2N, s, \text{R})$. We now wish to express the multiple summations in (122) in terms of appropriately defined contour integrals. In the case of a single variable this is straightforward, see Fig. 1 for a simple example. Let $f(\eta)$ be a meromorphic function with simple poles at the points $\eta = \theta_j$, and \mathcal{C}_{θ_j} a very small contour encircling θ_j . Residue calculus then gives

$$\sum_j \oint_{\mathcal{C}_{\theta_j}} \frac{d\eta}{2\pi} f(\eta) = \oint_{\mathcal{C}_{\text{tot}}} \frac{d\eta}{2\pi} f(\eta) - \sum_{i=1}^{N_{\mathcal{R}}} \oint_{\mathcal{C}_{\tilde{\theta}_i}} \frac{d\eta}{2\pi} f(\eta) \quad (123)$$

where \mathcal{C}_{tot} is a contour encircling the region \mathcal{R} , which contains all points θ_j as well as simple poles of $f(\eta)$ at positions $\tilde{\theta}_1, \dots, \tilde{\theta}_{N_{\mathcal{R}}}$.

The case of N integration variables η_1, \dots, η_N is more involved. Our function of interest is

$$G^{(s, \tilde{s})}(\eta_1, \dots, \eta_N) \equiv \frac{\bar{\rho}_N^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N)}{\rho_{2N}^{\tilde{s}}(-\tilde{\theta}_1, \dots, \tilde{\theta}_N)} \frac{\mathcal{K}_{2N}^s(\eta_1, \dots, \eta_N)}{\mathcal{K}_{2N}^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N)} \frac{f^{\beta/2}(\eta_N + i\pi, \dots, \tilde{\theta}_N)_{s, \tilde{s}}}{\prod_{k=1}^N \{e^{iQ_k^s(\eta_1, \dots, \eta_N)} - 1\}} e^{2i\Delta t \sum_{i=1}^N [\cosh \eta_i - \cosh \tilde{\theta}_i]}. \quad (124)$$

We first consider a contour \mathcal{C}_{tot} , which is the N -fold tensor product $\mathcal{C}(\mathbb{R}^+) \otimes \dots \otimes \mathcal{C}(\mathbb{R}^+)$, where $\mathcal{C}(\mathbb{R}^+)$ is a contour encircling \mathbb{R}^+ . Introducing $\mathcal{C}_{\tilde{\theta}_i}$ as the contour encircling $\tilde{\theta}_i$, we can express \mathcal{C}_{tot} in the form

$$\mathcal{C}_{\text{tot}} = \mathcal{G} \otimes \dots \otimes \mathcal{G} - \sum_{m=1}^N (-1)^m \sum_{1 \leq j_1 < \dots < j_m \leq N} \sum_{i_1, \dots, i_m=1}^N \mathcal{C}_{i_1, \dots, i_m}^{j_1, \dots, j_m}, \quad (125)$$

where \mathcal{G} is defined as

$$\mathcal{G} \equiv \mathcal{C}(\mathbb{R}^+) - \sum_{i=1}^N \mathcal{C}_{\tilde{\theta}_i}, \quad (126)$$

and $\mathcal{C}_{i_1, \dots, i_m}^{j_1, \dots, j_m}$ is defined as the multi-contour obtained from \mathcal{C}_{tot} by replacing the components j_1, \dots, j_m in the tensor product by $\mathcal{C}_{\tilde{\theta}_{i_1}}, \dots, \mathcal{C}_{\tilde{\theta}_{i_m}}$, i.e.

$$\mathcal{C}_{i_1, \dots, i_m}^{j_1, \dots, j_m} = \otimes_{j=1}^{j_1-1} \mathcal{C}(\mathbb{R}^+) \otimes \mathcal{C}_{\tilde{\theta}_{i_1}} \otimes_{j=j_1+1}^{j_2-1} \mathcal{C}(\mathbb{R}^+) \otimes \mathcal{C}_{\tilde{\theta}_{i_2}} \otimes_{j=j_2+1}^{j_3-1} \mathcal{C}(\mathbb{R}^+) \otimes \dots \otimes_{j=j_{m-1}+1}^{j_m-1} \mathcal{C}(\mathbb{R}^+) \otimes \mathcal{C}_{\tilde{\theta}_{i_m}} \otimes_{j=j_m+1}^N \mathcal{C}(\mathbb{R}^+). \quad (127)$$

We may use (125) to rewrite the integral of $G^{(s, \tilde{s})}$ over the multi-contour \mathcal{C}_{tot} as

$$\begin{aligned} \oint_{\mathcal{C}_{\text{tot}}} \prod_{i=1}^N \frac{d\eta_i}{2\pi} G^{(s, \tilde{s})}(\eta_1, \dots, \eta_N) &= \oint_{\mathcal{G} \otimes \dots \otimes \mathcal{G}} \prod_{i=1}^N \frac{d\eta_i}{2\pi} G^{(s, \tilde{s})}(\eta_1, \dots, \eta_N) \\ &\quad - \sum_{m=1}^N (-1)^m \sum_{1 \leq j_1 < \dots < j_m \leq N} \sum_{i_1, \dots, i_m=1}^N \oint_{\mathcal{C}_{i_1, \dots, i_m}^{j_1, \dots, j_m}} \prod_{i=1}^N \frac{d\eta_i}{2\pi} G^{(s, \tilde{s})}(\eta_1, \dots, \eta_N). \end{aligned} \quad (128)$$

Let us now focus on the first term in the right-hand side of (128). As the singularities of the form factor $f^{\beta/2}$ lie outside the contour $\mathcal{G} \otimes \dots \otimes \mathcal{G}$ and all other functions are expected to be well behaved, the only contributions to the integral arise from the regions characterized by

$$e^{i\tilde{Q}_k^s(\eta_1, \dots, \eta_N)} - 1 \approx 0, \quad k = 1, \dots, N. \quad (129)$$

By choosing $\mathcal{C}(\mathbb{R}^+)$ sufficiently close to the real axis, we can ensure that (129) is fulfilled only when $\{\eta_k\}$ is a real solution of the Bethe-Yang equations in the sector $(2N, s, \text{R})$ for some set $\{I_k\}$. We can ensure that *all* solutions to the Bethe-Yang equations in the sector $(2N, s, \text{R})$ are enclosed in $\mathcal{G} \otimes \dots \otimes \mathcal{G}$ by choosing the contours $\mathcal{C}_{\tilde{\theta}_i}$ to be sufficiently close to $\tilde{\theta}_i$, because $\{\tilde{\theta}_k\}$ is a solution of the Bethe-Yang equations in the NS sector and therefore cannot be arbitrarily close to any solution in the R sector. This allows us to conclude that

$$\oint_{\mathcal{G} \otimes \dots \otimes \mathcal{G}} \prod_{i=1}^N \frac{d\eta_i}{2\pi} G^{(s, \tilde{s})}(\eta_1, \dots, \eta_N) = \sum_{I_1, \dots, I_N \in \mathcal{C}_{\{\theta_\ell\}}} \oint \prod_{i=1}^N \frac{d\eta_i}{2\pi} G^{(s, \tilde{s})}(\eta_1, \dots, \eta_N). \quad (130)$$

Let us now turn to the contributions with $m = N$ on the right-hand side of Eq. (128). For these the relevant multi-contours are of the form

$$\mathcal{C}_{\tilde{\theta}_{\sigma(1)}} \otimes \dots \otimes \mathcal{C}_{\tilde{\theta}_{\sigma(N)}}, \quad (131)$$

where $\sigma \in \mathcal{S}_N$ is a permutation. Exploiting again the fact that $\{\tilde{\theta}_k\}$ is a solution of the Bethe-Yang equations in the NS sector, we may conclude that for contours $\mathcal{C}_{\tilde{\theta}_i}$ sufficiently close to the points $\tilde{\theta}_i$ we have for any point η_1, \dots, η_N inside the multi-contour

$$e^{i\tilde{Q}_k^s(\eta_1, \dots, \eta_N)} - 1 \neq 0, \quad \forall k. \quad (132)$$

Hence the only contributions to the integral arise from the annihilation poles of the form factor $f^{\beta/2}$. These considerations will be very useful in the following.

Substituting (130) into (128) we obtain the desired N -variable generalisation of (123)

$$\begin{aligned} \sum_{I_1, \dots, I_N \in \mathcal{C}_{\{\theta_\ell\}}} \oint \prod_{i=1}^N \frac{d\eta_i}{2\pi} G^{(s, \tilde{s})}(\eta_1, \dots, \eta_N) &= \oint_{\mathcal{C}_{\text{tot}}} \prod_{i=1}^N \frac{d\eta_i}{2\pi} G^{(s, \tilde{s})}(\eta_1, \dots, \eta_N) \\ &\quad + \sum_{m=1}^N (-1)^m \sum_{1 \leq j_1 < \dots < j_m \leq N} \sum_{i_1, \dots, i_m=1}^N \oint_{\mathcal{C}_{i_1, \dots, i_m}^{j_1, \dots, j_m}} \prod_{i=1}^N \frac{d\eta_i}{2\pi} G^{(s, \tilde{s})}(\eta_1, \dots, \eta_N). \end{aligned} \quad (133)$$

In order to deal with Eq. (122) we require one more step: since in (122) we are considering a fixed order of integers $\{I_k\}$, the corresponding solutions of the Bethe-Yang equations must satisfy $0 < \theta_1 < \dots < \theta_N$. In order to accommodate this ordering of rapidities we need to modify our multi-contours such that the i -th component depends on the variable z_{i+1} . For example, the integral on \mathcal{C}_{tot} changes to

$$\oint_{\mathcal{D}} \frac{dz_N}{2\pi} \oint_{\mathcal{D}(z_N)} \frac{dz_{N-1}}{2\pi} \dots \oint_{\mathcal{D}(z_2)} \frac{dz_1}{2\pi} f(z_1, \dots, z_N) \equiv \oint_{\tilde{\mathcal{C}}_{\text{tot}}} \prod_{i=1}^N \frac{dz_i}{2\pi} f(z_1, \dots, z_N), \quad (134)$$

where the contour $\mathcal{D}(z_N)$ encircles the interval $[0, \Re z_N]$. Reduced multi-contours $\tilde{\mathcal{C}}_{i_1, \dots, i_m}^{j_1, \dots, j_m}$ are defined in a way analogous to $\mathcal{C}_{i_1, \dots, i_m}^{j_1, \dots, j_m}$, but include additional constraints ensuring the ordering $\Re(z_N) > \Re(z_{N-1}) > \dots > \Re(z_1)$. We may now use these considerations to turn the sums in (122) into integrals over appropriate multi-contours in the form

$$\begin{aligned} \frac{\text{R}\langle \Psi_0 | e^{i\beta\Phi(t,x)/2} | \Phi_{\tilde{s}} \rangle_{\text{NS}}}{\text{NS}\langle \Psi_0 | \Phi_{\tilde{s}} \rangle_{\text{NS}}} &\approx \sum_s \oint_{\tilde{\mathcal{C}}_{\text{tot}}} \prod_{i=1}^N \frac{d\eta_i}{2\pi} G^{(s, \tilde{s})}(\eta_1, \dots, \eta_N) \\ &+ \sum_s \sum_{m=1}^{N-1} (-1)^m \sum_{1 \leq j_1 < \dots < j_m \leq N} \sum_{i_1, \dots, i_m=1}^N \oint_{\tilde{\mathcal{C}}_{i_1, \dots, i_m}^{j_1, \dots, j_m}} \prod_{i=1}^N \frac{d\eta_i}{2\pi} G^{(s, \tilde{s})}(\eta_1, \dots, \eta_n) \\ &+ (-1)^N \sum_s \oint_{\mathcal{C}_{\tilde{\theta}_1} \otimes \dots \otimes \mathcal{C}_{\tilde{\theta}_N}} \prod_{i=1}^N \frac{d\eta_i}{2\pi} G^{(s, \tilde{s})}(\eta_1, \dots, \eta_N). \end{aligned} \quad (135)$$

The crucial simplification occurring at this stage is that in the thermodynamic limit $L \rightarrow \infty$ and at late times we need to retain only the last term, as it features the most singular contribution, which occurs when⁴⁸

$$\eta_j \approx \tilde{\theta}_j, \quad j = 1, \dots, N. \quad (136)$$

As we have argued above, the only singularities we need to consider in the last term of (135) are those arising from the form factor. The *leading* singularities of the form factors can be calculated using the annihilation pole axiom, see Appendix F, and gives

$$f^{\beta/2}(\eta_N + i\pi, -\eta_N + i\pi, \dots, -\tilde{\theta}_N, \tilde{\theta}_N)_{s, \tilde{s}} \Big|_{\tilde{\theta}_j \approx \eta_j} = \mathcal{G}_{\beta/2} \delta_{\tilde{s}, s} \prod_{i=1}^N \frac{4}{(\eta_i - \tilde{\theta}_i)^2} + \text{less singular}. \quad (137)$$

An explicit expression for the vacuum expectation value $\mathcal{G}_{\beta/2} \equiv \langle 0 | e^{i\beta\Phi/2} | 0 \rangle$ is given in Ref. [34]. Substituting the expression (137) into the last term on the right-hand side of Eq. (135) gives the leading contribution at late times. Carrying out the contour integrals we obtain

$$\frac{\text{R}\langle \Psi_0 | e^{i\beta\Phi(t,x)/2} | \Phi_{\tilde{s}} \rangle_{\text{NS}}}{\text{NS}\langle \Psi_0 | \Phi_{\tilde{s}} \rangle_{\text{NS}}} \approx (-4i)^N \mathcal{G}_{\beta/2} \frac{\bar{\rho}_N^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N)}{\rho_{2N}^{\tilde{s}}(-\tilde{\theta}_1, \dots, \tilde{\theta}_N)} \partial_{\eta_1} \Big|_{\eta_1 = \tilde{\theta}_1} \dots \partial_{\eta_N} \Big|_{\eta_N = \tilde{\theta}_N} \prod_{i=1}^N \frac{e^{2i\Delta t [\cosh \eta_i - \cosh \tilde{\theta}_i]}}{e^{i\bar{Q}_i^{\tilde{s}}(\eta_1, \dots, \eta_N)} - 1} + \dots \quad (138)$$

The derivatives can be computed in leading order in the density δ using that $\partial_j \bar{Q}_j^{\tilde{s}} = \mathcal{O}(\ell)$, while $\partial_j \bar{Q}_i^{\tilde{s}} = \mathcal{O}(1)$ (see Appendix D)

$$\begin{aligned} \partial_{\eta_1} \Big|_{\eta_1 = \tilde{\theta}_1} \dots \partial_{\eta_N} \Big|_{\eta_N = \tilde{\theta}_N} \prod_{i=1}^N \frac{e^{2i\Delta t [\cosh \eta_i - \cosh \tilde{\theta}_i]}}{e^{i\bar{Q}_i^{\tilde{s}}(\eta_1, \dots, \eta_N)} - 1} &= \left(\frac{i}{4}\right)^N \prod_{i=1}^N \left\{ \partial_i \bar{Q}_i^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N) - 4\Delta t \sinh \tilde{\theta}_i \right\} \\ &+ \text{higher order in } \delta. \end{aligned} \quad (139)$$

Here we have used that

$$e^{i\bar{Q}_i^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N)} = -1. \quad (140)$$

The densities of states $\bar{\rho}_N^{\tilde{s}}$ and $\rho_{2N}^{\tilde{s}}$ are defined in terms of partial derivatives of the functions $\bar{Q}_j^{\tilde{s}}$ and $Q_j^{\tilde{s}}$, see Eqs. (84) and (57); their ratio can be calculated to leading order in the density δ by using the results derived in Appendix D as well

$$\frac{\bar{\rho}_N^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N)}{\rho_{2N}^{\tilde{s}}(-\tilde{\theta}_1, \dots, \tilde{\theta}_N)} = \prod_{i=1}^N \frac{1}{\partial_i \bar{Q}_i^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N)} + \text{higher order in } \delta. \quad (141)$$

Putting everything together, we obtain

$$\begin{aligned} \frac{\text{R}\langle\Psi_0|e^{i\beta\Phi(t,x)/2}|\Phi_{\tilde{s}}\rangle_{\text{NS}}}{\text{NS}\langle\Psi_0|\Phi_{\tilde{s}}\rangle_{\text{NS}}} &\approx \mathcal{G}_{\beta/2} \prod_{i=1}^N \left\{ 1 - \frac{4\Delta t \sinh \tilde{\theta}_i}{\partial_i \bar{Q}_i^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N)} \right\} + \dots \\ &= \mathcal{G}_{\beta/2} \exp \left[\sum_{i=1}^N \log \left(1 - \frac{4\Delta t \sinh \tilde{\theta}_i}{\partial_i \bar{Q}_i^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N)} \right) \right] + \dots \end{aligned} \quad (142)$$

Since $\partial_i \bar{Q}_i^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N) = \mathcal{O}(\ell) \gg \Delta t$, we may expand the logarithm

$$\sum_{i=1}^N \log \left(1 - \frac{4\Delta t \sinh \tilde{\theta}_i}{\partial_i \bar{Q}_i^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N)} \right) = -\frac{4vt}{L} \left\{ \sum_{i=1}^N \tanh \tilde{\theta}_i \right\} + \dots, \quad (143)$$

where we used

$$\frac{4\Delta t \sinh \tilde{\theta}_i}{\partial_i \bar{Q}_i^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N)} = \frac{4\Delta t}{\ell} \tanh \tilde{\theta}_i + \text{higher order in } \delta. \quad (144)$$

Finally, we convert the sum over $\tilde{\theta}_j$ into an integral in the $L \rightarrow \infty$ limit

$$\begin{aligned} \lim_{L \rightarrow \infty} \frac{\text{R}\langle\Psi_0|e^{i\beta\Phi(t,x)/2}|\Phi_{\tilde{s}}\rangle_{\text{NS}}}{\text{NS}\langle\Psi_0|\Phi_{\tilde{s}}\rangle_{\text{NS}}} &\approx \lim_{L \rightarrow \infty} \mathcal{G}_{\beta/2} \exp \left[-\frac{4vt}{L} \sum_{i=1}^N \tanh \tilde{\theta}_i \right] + \dots \\ &= \mathcal{G}_{\beta/2} \exp \left[-4vt \int_0^\infty d\theta \rho_\Phi(\theta) \tanh \theta \right] + \dots \\ &= \mathcal{G}_{\beta/2} e^{-t/\tau} + \dots \end{aligned} \quad (145)$$

The decay time τ defined in this way is given by

$$\tau^{-1} \equiv \frac{2\Delta}{\pi} \int_0^\infty d\theta [G(\theta) \sinh \theta + \mathcal{O}(K^4)]. \quad (146)$$

Using $\mathcal{G}_a = \mathcal{G}_{-a} = \mathcal{G}_a^*$, we obtain our final result for the time evolution of the one-point function in the thermodynamic limit at late times $\Delta t \gg 1$, and in the low-density limit $\delta \ll 1$

$$\lim_{L \rightarrow \infty} \langle e^{i\beta\Phi(t,x)/2} \rangle_L = \lim_{L \rightarrow \infty} \frac{L \langle \Psi_t | e^{i\beta\Phi(x)/2} | \Psi_t \rangle_L}{L \langle \Psi_t | \Psi_t \rangle_L} = \mathcal{G}_{\beta/2} e^{-t/\tau} + \dots \quad (147)$$

In the next section we show that this result can also be obtained using a complementary approach.

VI. LINKED-CLUSTER EXPANSION APPROACH

A characteristic feature of initial states after quantum quenches is that they involve finite densities of excitations of the post-quench Hamiltonian. In this sense the quench problem is similar to the one of finite-temperature dynamics in equilibrium. Building on recent progress in the latter problem^{49,50}, Refs. 5 and 17 developed an approach to quench problems based on a linked-cluster expansion for initial states of the form (32). Here we generalize this approach to the case of the sine-Gordon model.

A. Formal expansion of one-point functions

Our goal is the derivation of a formal expansion of the one-point function

$$\frac{\langle \Psi_t | \mathcal{O} | \Psi_t \rangle}{\langle \Psi_t | \Psi_t \rangle} = \frac{\langle \Psi_0 | e^{i\mathcal{H}_{\text{SG}} t} \mathcal{O} e^{-i\mathcal{H}_{\text{SG}} t} | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} \quad (148)$$

in powers of the quench matrix K^{ab} , which physically corresponds to an expansion in the density of post-quench excitations. The operator \mathcal{O} is assumed to be semi-local with respect to the fundamental field creating solitons and

antisolitons, and to possess Lorentz spin s . This is reflected in its form factors and the axioms which they must satisfy. For completeness we state them in App. A. For the operator $\mathcal{O} = e^{i\alpha\Phi}$ one has $l_{\pm}^{\alpha} = e^{\pm 2\pi i\alpha/\beta}$ and Lorentz spin $s = 0$. Thus for $\alpha = \beta/2$, the semi-locality factor simplifies to $l_a^{\beta/2} = -1$ for both solitons and antisolitons. The initial state $|\Psi_0\rangle$ is assumed to be of the form (32) with a suitable ultra-violet regularisation provided for example by introducing an extrapolation time (36).

First we expand the initial states in both the numerator and denominator of (148) to obtain the formal expansions

$$\langle \Psi_t | \mathcal{O} | \Psi_t \rangle = \sum_{M,N=0}^{\infty} \int_0^{\infty} \frac{d\theta'_1 \dots d\theta'_M}{M!(2\pi)^M} \frac{d\theta_1 \dots d\theta_N}{N!(2\pi)^N} \prod_{i=1}^M (K^{a_i b_i}(\theta'_i))^* \prod_{j=1}^N K^{c_j d_j}(\theta_j) e^{2\Delta i t \sum_i \cosh \theta'_i} e^{-2\Delta i t \sum_j \cosh \theta_j} \\ \times_{b_1 a_1 \dots b_M a_M} \langle \theta'_1, -\theta'_1, \dots, \theta'_M, -\theta'_M | \mathcal{O} | -\theta_N, \theta_N, \dots, -\theta_1, \theta_1 \rangle_{c_N d_N \dots c_1 d_1} \quad (149)$$

$$\equiv \sum_{M,N=0}^{\infty} C_{2M,2N}(t) \quad (150)$$

as well as

$$\langle \Psi_0 | \Psi_0 \rangle = 1 + \sum_{N=1}^{\infty} \int_0^{\infty} \frac{d\theta'_1 \dots d\theta'_N}{N!(2\pi)^N} \frac{d\theta_1 \dots d\theta_N}{N!(2\pi)^N} \prod_{i=1}^N (K^{a_i b_i}(\theta'_i))^* K^{c_i d_i}(\theta_i) \\ \times_{b_1 a_1 \dots b_N a_N} \langle \theta'_1, -\theta'_1, \dots, \theta'_N, -\theta'_N | -\theta_N, \theta_N, \dots, -\theta_1, \theta_1 \rangle_{c_N d_N \dots c_1 d_1} \quad (151)$$

$$\equiv \sum_{N=0}^{\infty} Z_{2N}. \quad (152)$$

Note that the indices $2M$ and $2N$ correspond to the number of particles originating from the left and right initial state respectively. Furthermore, here and in the following we implicitly sum over all indices $a_i, b_i, c_i, d_i = \pm$ appearing twice. The expansion (152) implies

$$\frac{1}{\langle \Psi_0 | \Psi_0 \rangle} = 1 - Z_2 + Z_2^2 - Z_4 + \mathcal{O}(K^6). \quad (153)$$

We note that (153) defines linked clusters, i.e. it identifies the parts of the numerator in (148) that diverge in the infinite volume.

The matrix elements in the terms $C_{2M,2N}(t)$ and Z_{2N} possess kinematical poles whenever $\theta'_i = \theta_j$ and therefore have to be regularized. This is done following Smirnov³² by shifting the rapidities of the, say, outgoing states into the complex plane as outlined in detail in Appendix G. After this regularisation the denominator and numerator in (148) still contain divergencies due to the intertwining of particles with rapidities θ_i and $-\theta_i$ in the initial state (32). These divergences are a consequence of working in the infinite volume and have to be canceled against each other. In order to exhibit these cancellations explicitly we use the κ -regularisation scheme originally introduced in the study of finite-temperature correlation functions⁴⁹ and recently generalized to the quench dynamics of Ising systems^{5,17}. The details of this are given in Appendix H.

Combining these two regularisations the expansions (150) and (152) are finite but may contain terms proportional to the system size, $\propto L^k$, that diverge in the infinite-volume limit. However, considering their ratio and expanding again formally in powers of the quench matrix we find

$$\frac{\langle \Psi_t | \mathcal{O} | \Psi_t \rangle}{\langle \Psi_t | \Psi_t \rangle} = \frac{\sum_{M,N=0}^{\infty} C_{2M,2N}(t)}{\sum_{N=0}^{\infty} Z_{2N}} \equiv \sum_{M,N=0}^{\infty} D_{2M,2N}(t), \quad (154)$$

where all terms $\propto L^k$ with $k \geq 1$ cancel each other and the remaining functions $D_{2M,2N}(t)$ are finite in the infinite-volume limit. This is analogous to a linked-cluster expansion of finite-temperature correlators^{49,50}. In the following we calculate the first few terms of this expansion, specifically the leading terms up to $\mathcal{O}(K^4)$ in the expansion (154) for the operator $\mathcal{O} = e^{i\beta\Phi/2}$, and the terms up to $\mathcal{O}(K^2)$ for the more general operator $\mathcal{O} = e^{i\alpha\Phi}$ in Section VIE.

B. Terms in $\mathcal{O}(K^0)$ and $\mathcal{O}(K)$

The terms in these orders do not contain any divergencies and can be straightforwardly calculated with the results

$$D_{00}(t) = C_{00}(t) = \mathcal{G}_{\beta/2}, \quad (155)$$

$$D_{20}(t) = C_{20}(t) = \int_0^\infty \frac{d\theta}{2\pi} (K^{ab}(\theta))^* f_{ab}^{-\beta/2}(-\theta, \theta) e^{2\Delta i t \cosh \theta}, \quad (156)$$

$$D_{02}(t) = C_{02}(t) = \int_0^\infty \frac{d\theta}{2\pi} K^{ab}(\theta) f_{ab}^{\beta/2}(-\theta, \theta) e^{-2\Delta i t \cosh \theta}. \quad (157)$$

Here $f_{ab}^{\pm\beta/2}(\theta_1, \theta_2) = \langle 0 | e^{\pm i\beta\Phi/2} | \theta_1, \theta_2 \rangle_{ab}$ is the two-particle form factor⁵¹ of $e^{\pm i\beta\Phi/2}$, which we state for completeness in Appendix I. The long-time behaviour of $D_{20}(t) + D_{02}(t)$ is given by

$$D_{20}(t) + D_{02}(t) \propto \frac{\cos(2\Delta t)}{(\Delta t)^{3/2}}. \quad (158)$$

C. Terms in $\mathcal{O}(K^2)$

In this order there exist three terms. The first two originate from $C_{40}(t)$ and $C_{04}(t)$, which are completely regular. These two terms simply yield sub-leading corrections to $D_{20}(t) + D_{02}(t)$ stated above and will not be considered in the following.

The third term, however, contains kinematical poles as well as infinite-volume divergencies. From the formal expansion (154) we deduce that

$$D_{22}(t) = C_{22}(t) - Z_2 C_{00}(t) = C_{22}(t) - Z_2 \mathcal{G}_{\beta/2}. \quad (159)$$

Here the first non-trivial term in the expansion (152) reads using the infinite-volume regularisation discussed in Appendix H

$$Z_2 \equiv \int d\kappa P(\kappa) \int_0^\infty \frac{d\theta' d\theta}{(2\pi)^2} (K^{ab}(\theta'))^* K^{cd}(\theta) {}_{ba}\langle \theta', -\theta' | -\theta + \kappa, \theta + \kappa \rangle_{cd} \quad (160)$$

$$= \int d\kappa P(\kappa) \delta(-2\kappa) \int_0^\infty d\theta (K^{ab}(\theta + \kappa))^* K^{ab}(\theta) = \frac{L}{2} \int_0^\infty d\theta G(\theta), \quad (161)$$

where we implicitly sum over $a, b = \pm$ and used the notation (111), i.e. $G(\theta) = |K^{ab}(\theta)|^2$. The infinite-volume divergence is now clearly exhibited.

On the other hand, we have to consider

$$C_{22}(t) = \int_0^\infty \frac{d\theta' d\theta}{(2\pi)^2} (K^{ab}(\theta'))^* K^{cd}(\theta) {}_{ba}\langle \theta', -\theta' | e^{i\beta\Phi/2} | -\theta, \theta \rangle_{cd} e^{2\Delta i t (\cosh \theta' - \cosh \theta)}. \quad (162)$$

Again shifting the rapidities $\pm\theta$ by the auxiliary parameter κ and regularising the form factor by analytically continuing in the rapidities $\pm\theta'$ as discussed in Appendix G we obtain

$${}_{ba}\langle \theta', -\theta' | e^{i\beta\Phi/2} | -\theta, \theta \rangle_{cd} = {}_{ba}\langle \theta', -\theta' | e^{i\beta\Phi/2} | -\theta + \kappa, \theta + \kappa \rangle_{cd} \quad (163)$$

$$= (2\pi)^2 \mathcal{G}_{\beta/2} \delta_a^c \delta_b^d \delta(-2\kappa) \delta(\theta' - \theta + \kappa) \quad (164)$$

$$+ 2\pi S_{ba}^{ef} (2\theta - 2\kappa) S_{cd}^{fh} (-2\theta) \delta(\theta' - \theta + \kappa) {}_e\langle \theta' + i0 | e^{i\beta\Phi/2} | \theta + \kappa \rangle_h \quad (165)$$

$$+ 2\pi \delta_b^d \delta(\theta' - \theta - \kappa) {}_a\langle -\theta' + i0 | e^{i\beta\Phi/2} | -\theta + \kappa \rangle_c \quad (166)$$

$$+ {}_{ba}\langle \theta' + i0, -\theta' + i0 | e^{i\beta\Phi/2} | -\theta + \kappa, \theta + \kappa \rangle_{cd}, \quad (167)$$

where we have used (24) and (26). Furthermore, we have dropped two contributions which result in expressions like $\delta(-\theta' - \theta - \kappa) \delta(\theta' + \theta - \kappa)$, since they yield only terms $\propto \kappa$ vanishing after multiplying with $P(\kappa)$ and integrating over κ (see Ref. 17 for a detailed discussion of similar terms for the Ising field theory).

Plugging (163) into (162) we obtain three different terms which can be thought of as being disconnected, semi-connected and fully connected. Specifically we have

$$C_{22}(t) = C_{22}^0(t) + C_{22}^1(t) + C_{22}^2(t). \quad (168)$$

In the following we will analyse each term separately.

1. *Disconnected contribution: C_{22}^0*

The fully disconnected contribution follows from (164). Explicitly we find

$$C_{22}^0(t) = \mathcal{G}_{\beta/2} \delta(-2\kappa) \int_0^\infty d\theta |K^{ab}(\theta)|^2 = \mathcal{G}_{\beta/2} \delta(-2\kappa) \int_0^\infty d\theta G(\theta). \quad (169)$$

Now multiplying with $P(\kappa)$ and integrating over κ we obtain

$$C_{22}^0(t) = \frac{L}{2} \mathcal{G}_{\beta/2} \int_0^\infty d\theta G(\theta) = Z_2 \mathcal{G}_{\beta/2}, \quad (170)$$

i.e. this cancels the second term in (159). The remainder of $D_{22}(t)$ does not contain terms $\propto \delta(\kappa)$.

2. *Semi-connected contribution: C_{22}^1*

Next consider the terms (165) and (166). We find

$$\begin{aligned} C_{22}^1(t) &= \int_0^\infty \frac{d\theta}{2\pi} (K^{ab}(\theta))^* K^{cd}(\theta + \kappa) S_{ba}^{ef}(2\theta) S_{cd}^{fh}(-2\theta - 2\kappa) {}_e\langle \theta + i0 | e^{i\beta\Phi/2} | \theta + 2\kappa \rangle_h e^{2\Delta it [\cosh \theta - \cosh(\theta + \kappa)]} \\ &\quad + \int_0^\infty \frac{d\theta}{2\pi} (K^{ab}(\theta))^* K^{cb}(\theta - \kappa) {}_a\langle -\theta + i0 | e^{i\beta\Phi/2} | -\theta + 2\kappa \rangle_c e^{2\Delta it [\cosh \theta - \cosh(\theta - \kappa)]} \end{aligned} \quad (171)$$

$$\begin{aligned} &= -f_{eh}^{\beta/2}(i\pi + i0, 2\kappa) \int_0^\infty \frac{d\theta}{2\pi} (K^{ab}(\theta))^* K^{cd}(\theta + \kappa) S_{ba}^{ef}(2\theta) S_{cd}^{fh}(-2\theta - 2\kappa) e^{2\Delta it [\cosh \theta - \cosh(\theta + \kappa)]} \\ &\quad - f_{ac}^{\beta/2}(i\pi + i0, 2\kappa) \int_0^\infty \frac{d\theta}{2\pi} (K^{ab}(\theta))^* K^{cb}(\theta - \kappa) e^{2\Delta it [\cosh \theta - \cosh(\theta - \kappa)]}. \end{aligned} \quad (172)$$

Now we use the annihilation pole axiom (see Appendix A)

$$f_{ab}^{\beta/2}(i\pi + i0, 2\kappa) = -f_{ba}^{\beta/2}(2\kappa + 2i\pi, i\pi + i0) = -2i \frac{C_{ab} \mathcal{G}_{\beta/2}}{2\kappa - i0} - F_{ba}^{\beta/2}(\kappa), \quad (173)$$

where $F_{ba}^{\beta/2}(\kappa)$ is analytic for $\kappa \rightarrow 0$. Finally we expand C_{22}^1 in κ and keep only the terms $\propto 1/\kappa$ and $\propto t$, since all terms $\propto \kappa^n$ with $n \geq 1$ vanish in the infinite-volume limit. We obtain

$$\begin{aligned} C_{22}^1(t) &= \frac{2i \mathcal{G}_{\beta/2}}{2\kappa - i0} \int_0^\infty \frac{d\theta}{2\pi} (K^{ab}(\theta))^* K^{cd}(\theta) S_{ab}^{fe}(2\theta) S_{fe}^{cd}(-2\theta) [1 - 2\Delta it \kappa \sinh \theta] \\ &\quad + \frac{2i \mathcal{G}_{\beta/2}}{2\kappa - i0} \int_0^\infty \frac{d\theta}{2\pi} |K^{ab}(\theta)|^2 [1 + 2\Delta it \kappa \sinh \theta] + \dots \end{aligned} \quad (174)$$

Obviously this contains static terms $\propto 1/\kappa$, i.e. there are still remnants of infinite-volume divergencies. As we are going to show in the next subsection, these terms will cancel against contributions from the fully connected piece C_{22}^2 .

3. *Fully connected contribution: C_{22}^2*

Finally let us consider the term (167). Plugging it into (162) yields

$$C_{22}^2(t) = \int_0^\infty \frac{d\theta' d\theta}{(2\pi)^2} (K^{ab}(\theta'))^* K^{cd}(\theta) f_{ba\bar{c}d}^{\beta/2}(\theta' + i\pi + i0, -\theta' + i\pi + i0, -\theta + \kappa, \theta + \kappa) e^{2\Delta it (\cosh \theta' - \cosh \theta)}. \quad (175)$$

The form factor possesses annihilation poles at the positions $\theta = \pm\theta' - \kappa + i0$ and $\theta = \mp\theta' + \kappa - i0$. We deal with these poles by shifting the contour of integration for θ to the lower half plane. Doing this we pick up a pole at $\theta = \theta' + \kappa - i0$ and obtain

$$C_{22}(t) = C_{22}'(t) + C_{22}^p(t), \quad (176)$$

where

$$C'_{22}(t) = \int_0^\infty \frac{d\theta'}{2\pi} \int_{\gamma_-} \frac{d\theta}{2\pi} (K^{ab}(\theta'))^* K^{cd}(\theta) f_{\bar{b}acd}^{\beta/2}(\theta' + i\pi + i0, -\theta' + i\pi + i0, -\theta + \kappa, \theta + \kappa) e^{2\Delta it(\cosh \theta' - \cosh \theta)} \quad (177)$$

with the contour of integration γ_- parametrized by $(0 < \phi_0 \leq \pi/4)$

$$\gamma_-(s) = \begin{cases} -is, & 0 \leq s \leq \phi_0, \\ (s - \phi_0) - i\phi_0, & \phi_0 \leq s < \infty. \end{cases} \quad (178)$$

and

$$C_{22}^p(t) = -i \int_0^\infty \frac{d\theta'}{2\pi} (K^{ab}(\theta'))^* K^{cd}(\theta' + \kappa) e^{2\Delta it[\cosh \theta' - \cosh(\theta' + \kappa)]} \\ \times \text{Res}[f_{\bar{b}acd}^{\beta/2}(\theta' + i\pi + i0, -\theta' + i\pi + i0, -\theta + \kappa, \theta + \kappa), \theta = \theta' + \kappa - i0]. \quad (179)$$

The leading long-time behaviour comes from the pole contribution $C_{22}^p(t)$, while $C'_{22}(t)$ yields a correction and will not be considered further. The corresponding term for the time evolution of the order parameter field in the Ising field theory gives rise to a correction $C'_{22}(t) \sim 1/t$ at late times¹⁷. In order to evaluate $C_{22}^p(t)$ we first have to determine the appearing residue. This is done using the annihilation pole axiom stated in Appendix A as well as $\text{Res}[f(z), z = z_0] = -\text{Res}[f(-z), z = -z_0]$, resulting in

$$\begin{aligned} & \text{Res}[f_{\bar{b}acd}^{\beta/2}(\theta' + i\pi + i0, -\theta' + i\pi + i0, -\theta + \kappa, \theta + \kappa), \theta = \theta' + \kappa - i0] \\ &= \text{Res}[S_{\bar{b}a}^{ef}(2\theta') S_{cd}^{gh}(-2\theta) f_{fegh}^{\beta/2}(-\theta' + i\pi + i0, \theta' + i\pi + i0, \theta + \kappa, -\theta + \kappa), \theta = \theta' + \kappa - i0] \\ &= -S_{\bar{b}a}^{ef}(2\theta') S_{cd}^{gh}(-2\theta' - 2\kappa) \text{Res}[f_{fegh}^{\beta/2}(-\theta + \kappa + 2i\pi, -\theta' + i\pi + i0, \theta' + i\pi + i0, \theta + \kappa), \theta = \theta' + \kappa - i0] \\ &= iC_{gk} S_{\bar{b}a}^{ef}(2\theta') S_{cd}^{gh}(-2\theta' - 2\kappa) f_{ij}^{\beta/2}(i\pi + i0, 2\kappa) [\delta_e^i \delta_h^j \delta_f^k + S_{fe}^{li}(-2\theta) S_{lh}^{kj}(-2\theta' - 2\kappa + i\pi)]. \end{aligned} \quad (180)$$

Hence we get

$$C_{22}^p(t) = C_{gk} f_{ij}^{\beta/2}(i\pi + i0, 2\kappa) \int_0^\infty \frac{d\theta}{2\pi} (K^{ab}(\theta))^* K^{cd}(\theta + \kappa) S_{\bar{b}a}^{ef}(2\theta) S_{cd}^{gh}(-2\theta - 2\kappa) \\ \times [\delta_e^i \delta_h^j \delta_f^k + S_{fe}^{li}(-2\theta) S_{lh}^{kj}(-2\theta - 2\kappa + i\pi)] e^{2\Delta it[\cosh \theta - \cosh(\theta + \kappa)]}, \quad (181)$$

which is further simplified by expanding in κ up to terms $\propto 1/\kappa$ and $\propto t$ and using (173). This yields

$$\begin{aligned} C_{22}^p(t) &= -i \frac{2\mathcal{G}_{\beta/2}}{2\kappa - i0} \int_0^\infty \frac{d\theta}{2\pi} (K^{ab}(\theta))^* K^{cd}(\theta) S_{\bar{b}a}^{ef}(2\theta) S_{cd}^{\bar{f}\bar{e}}(-2\theta) [1 - 2\Delta it\kappa \sinh \theta] \\ &\quad - i \frac{2\mathcal{G}_{\beta/2}}{2\kappa - i0} \int_0^\infty \frac{d\theta}{2\pi} (K^{ab}(\theta))^* K^{cd}(\theta) [1 - 2\Delta it\kappa \sinh \theta] \\ &\quad \times S_{\bar{b}a}^{ef}(2\theta) S_{cd}^{gh}(-2\theta) S_{fe}^{l\bar{j}}(-2\theta) S_{lh}^{\bar{g}j}(-2\theta + i\pi) + \dots \end{aligned} \quad (182)$$

Now using the crossing relation (25) as well as (26) we obtain

$$\begin{aligned} C_{22}^p(t) &= -i \frac{2\mathcal{G}_{\beta/2}}{2\kappa - i0} \int_0^\infty \frac{d\theta}{2\pi} (K^{ab}(\theta))^* K^{cd}(\theta) S_{ab}^{ef}(2\theta) S_{ef}^{cd}(-2\theta) [1 - 2\Delta it\kappa \sinh \theta] \\ &\quad - i \frac{2\mathcal{G}_{\beta/2}}{2\kappa - i0} \int_0^\infty \frac{d\theta}{2\pi} |K^{ab}(\theta)|^2 [1 - 2\Delta it\kappa \sinh \theta] + \dots \end{aligned} \quad (183)$$

4. Final result for D_{22}

We are now in a position to write down the final result for (159). As already discussed the terms $\propto Z_2$ cancel and we are left with

$$D_{22}(t) = C_{22}^1(t) + C_{22}^p(t) + C'_{22}(t). \quad (184)$$

The leading long-time behaviour is given by the first and second term which we will analyse in the following. In contrast, the last term stated in (177) constitutes a sub-leading correction and will not be considered further.

For the leading terms (174) and (183) we find that the ill-defined terms $\propto 1/\kappa$ as well as the terms containing scattering matrices cancel each other. We are thus left with

$$C_{22}^1(t) + C_{22}^p(t) = -\frac{2\kappa}{2\kappa - i0} 4\mathcal{G}_{\beta/2} \Delta t \int_0^\infty \frac{d\theta}{2\pi} G(\theta) \sinh \theta. \quad (185)$$

Now employing the κ -regularisation scheme using

$$\int d\kappa P(\kappa) \frac{2\kappa}{2\kappa - i0} = 1 \quad (186)$$

we obtain the final result

$$D_{22}(t) = -\mathcal{G}_{\beta/2} \frac{t}{\tau} + C_{22}'(t), \quad (187)$$

with the decay rate in $\mathcal{O}(K^2)$ given by

$$\tau^{-1} = \frac{2\Delta}{\pi} \int_0^\infty d\theta G(\theta) \sinh \theta, \quad (188)$$

which was already obtained in the representative state approach (146) with $G(\theta) = \sum_{ab} |K^{ab}(\theta)|^2$.

D. Leading long-time behaviour in $\mathcal{O}(K^4)$ and final result

As we have derived above, the contributions in $\mathcal{O}(K^2)$ grow linear in time, see Eq. (187). The same behaviour was observed in the linked-cluster expansion performed for Ising systems^{5,17}, where it was also shown that the leading behaviour in $\mathcal{O}(K^{2n})$ is given by terms growing as t^n at late times. While an analysis of the leading long-time behaviour at arbitrary order in the quench matrix is out of reach in the sine-Gordon model, we have extracted the long-time asymptotics at $\mathcal{O}(K^4)$. As we show in Appendix J it is given by

$$D_{44}(t) \sim C_{44}(t) = \frac{2\Delta^2 t^2}{\pi^2} \mathcal{G}_{\beta/2} \int_0^\infty d\theta_1 d\theta_2 G(\theta_1) G(\theta_2) \sinh \theta_1 \sinh \theta_2 + \dots = \frac{\mathcal{G}_{\beta/2}}{2} \left(\frac{t}{\tau} \right)^2 + \dots \quad (189)$$

In the derivation of (189) we have assumed the absence of diagonal terms in the quench matrix, i.e. $K^{aa}(\theta) = 0$ (this is the case for fixed boundary conditions). Furthermore, the derivation of Appendix J is restricted to terms quadratic in t , i.e. we are not able to extract corrections to the decay time τ in $\mathcal{O}(K^4)$.

Together with the results derived above we thus obtain for the leading behaviour up to $\mathcal{O}(K^4)$

$$\frac{\langle \Psi_t | e^{i\beta\Phi/2} | \Psi_t \rangle}{\langle \Psi_t | \Psi_t \rangle} = \mathcal{G}_{\beta/2} \left[1 - \frac{t}{\tau} + \frac{1}{2} \left(\frac{t}{\tau} \right)^2 + \dots \right] = \mathcal{G}_{\beta/2} e^{-t/\tau} (1 + \dots), \quad (190)$$

$$\tau^{-1} = \frac{2\Delta}{\pi} \int_0^\infty d\theta G(\theta) \sinh \theta + \mathcal{O}(K^4). \quad (191)$$

As we clearly see the leading terms up to the order considered here are consistent with an exponential decay of the one-point function also deduced from the representative-state approach in Eq. (147). The dots represent corrections which are, up to $\mathcal{O}(K^2)$ given by $D_{20}(t) + D_{02}(t) \propto \cos(2\Delta t)/(\Delta t)^{3/2}$, $D_{40}(t) + D_{04}(t)$ and $C_{22}'(t)$.

E. Time evolution for $\alpha \neq \beta/2$

The calculation of the time evolution of $e^{i\alpha\Phi}$ for $\alpha \neq \beta/2$ is significantly more involved since the semi-locality factor l_a^α now depends on the index a . Up to $\mathcal{O}(K)$ this complications do not show up and the results are simply given by the ones of Section VIB with the normalisation³⁴ \mathcal{G}_α and two-particle form factors³⁵ $f_{ab}^\alpha(\theta_1, \theta_2)$.

In $\mathcal{O}(K^2)$, however, there appear more pronounced changes. Starting with (162) the form factor can still be regularized using (164)–(167) yielding a disconnected, semi-connected and fully connected contribution. The first is essentially unchanged, $C_{22}^0(t) = Z_2 \mathcal{G}_\alpha$. Differences appear in C_{22}^1 due to the non-trivial semi-locality factors in the crossing relation (G11) and the annihilation pole axiom. We find after using

$$f_{ab}^\alpha(i\pi + i0, 2\kappa) = l_b^\alpha f_{ba}^\alpha(2\kappa + 2i\pi, i\pi + i0) = i(l_b^\alpha - 1) \frac{C_{ab} \mathcal{G}_\alpha}{2\kappa - i0} + l_b^\alpha F_{ba}(\kappa), \quad (192)$$

where $F_{ba}(\kappa)$ is again analytic for $\kappa \rightarrow 0$, and expanding in κ keeping only the terms $\propto 1/\kappa$ and $\propto t$

$$\begin{aligned} C_{22}^1(t) &= i(1 - l_e^\alpha) \frac{\mathcal{G}_\alpha}{2\kappa - i0} \int_0^\infty \frac{d\theta}{2\pi} (K^{ab}(\theta))^* K^{cd}(\theta) S_{ab}^{fe}(2\theta) S_{fe}^{cd}(-2\theta) [1 - 2\Delta i t \kappa \sinh \theta] \\ &\quad + i(1 - l_a^\alpha) \frac{\mathcal{G}_\alpha}{2\kappa - i0} \int_0^\infty \frac{d\theta}{2\pi} |K^{ab}(\theta)|^2 [1 + 2\Delta i t \kappa \sinh \theta] + \dots \end{aligned} \quad (193)$$

For the fully connected contribution we again have to determine the residue, which is now given by

$$\begin{aligned} \text{Res}[f_{\bar{b}acd}^\alpha(\theta' + i\pi + i0, -\theta' + i\pi + i0, -\theta + \kappa, \theta + \kappa), \theta = \theta' + \kappa - i0] \\ = -iC_{gk} l_g^\alpha S_{\bar{b}a}^{ef}(2\theta') S_{cd}^{gh}(-2\theta' - 2\kappa) f_{ij}^\alpha(i\pi + i0, 2\kappa) [\delta_e^i \delta_h^j \delta_f^k - l_g^\alpha S_{fe}^{li}(-2\theta') S_{lh}^{kj}(-2\theta' - 2\kappa + i\pi)]. \end{aligned} \quad (194)$$

Now performing the same steps as previously we obtain for the expansion in κ

$$C_{22}^p(t) = -il_a^\alpha l_b^\alpha l_{\bar{e}}^\alpha (l_f^\alpha - 1) \frac{\mathcal{G}_\alpha}{2\kappa - i0} \int_0^\infty \frac{d\theta}{2\pi} (K^{ab}(\theta))^* K^{cd}(\theta) S_{ab}^{ef}(2\theta) S_{ef}^{cd}(-2\theta) [1 - 2\Delta i t \kappa \sinh \theta] \quad (195)$$

$$+ il_a^\alpha l_b^\alpha (l_b^\alpha - 1) \frac{\mathcal{G}_\alpha}{2\kappa - i0} \int_0^\infty \frac{d\theta}{2\pi} |K^{ab}(\theta)|^2 [1 - 2\Delta i t \kappa \sinh \theta] + \dots \quad (196)$$

Summing these two contributions we see that the terms $\propto 1/\kappa$ cancel, i.e. the result in this order is well-defined in the infinite-volume limit, and we obtain

$$D_{22}(t) = -\mathcal{G}_\alpha \frac{t}{\tau_\alpha}, \quad \tau_\alpha^{-1} = \Delta \int_0^\infty \frac{d\theta}{2\pi} [|K^{aa}(\theta)|^2 (1 - l_a^\alpha)^2 + 2|K^{a\bar{a}}(\theta)|^2 (1 - l_a^\alpha)] \sinh \theta. \quad (197)$$

For $\alpha = \beta/2$ this simplifies to our previous result (188). However, for the local operator $e^{i\beta\Phi}$ studied by Gritsev et al.²⁰ the contribution (197) vanishes, and our results do not provide information on the time evolution of $F_{\Psi_0}^\beta(t)$.

For initial states fulfilling (41) (the state corresponding to Dirichlet boundary conditions being an example) this can be simplified further to

$$\tau_\alpha^{-1} = \frac{4\Delta}{\pi} \sin^2\left(\frac{\pi\alpha}{\beta}\right) \int_0^\infty d\theta |K(\theta)|^2 \sinh \theta, \quad (198)$$

where $K(\theta) = K^{a\bar{a}}(\theta) = K^{\bar{a}a}(\theta)$ is independent of a .

VII. SUMMARY AND CONCLUSIONS

We have considered the time evolution of the semi-local operator $e^{i\beta\Phi/2}$ after an “integrable” quench to the sine-Gordon model. The system was assumed to be initialised in an integrable boundary state of the form

$$|\Psi_0\rangle = \exp\left(\int_0^\infty \frac{d\theta}{2\pi} K^{ab}(\theta) Z_a^\dagger(-\theta) Z_b^\dagger(\theta)\right) |0\rangle, \quad (199)$$

and we focussed on the behaviour of the expectation value $\langle\Psi_0|e^{i\beta\Phi(t)/2}|\Psi_0\rangle/\langle\Psi_0|\Psi_0\rangle$ at late times after the quench. We developed two novel methods to evaluate this expectation value for the case of a “small” quench, defined as involving small densities of excitations of the post-quench Hamiltonian. The first approach is based on the concept of a *representative state*¹⁶, while the second entails the generalisation of linked cluster expansions first developed for studying finite-temperature dynamics in integrable models^{49,50} to integrable models with non-trivial scattering out of equilibrium. Our main result is that in the thermodynamic limit at late times $\Delta t \gg 1$, and for low densities $\delta \ll 1$

$$\frac{\langle\Psi_0|e^{i\beta\Phi(x,t)/2}|\Psi_0\rangle}{\langle\Psi_0|\Psi_0\rangle} = \mathcal{G}_{\beta/2} e^{-t/\tau} + \dots, \quad (200)$$

where the decay time τ is given in Eq. (146) or (191). We note that exponential decay has also been observed in numerical simulations⁵² of the staggered magnetisation after interaction quenches in the XXZ Heisenberg chain.

To the best of our knowledge we provide here the first successful analytic calculation of the time dependence of a local observable for a quench to an *interacting* integrable model. In order to be able to carry out our calculations we required several simplifying assumptions:

1. the initial state was taken to be of the form (199);
2. we required the functions $K^{ab}(\theta)$ to be uniformly small, which corresponds to the limit of low densities of excitations after the quench;
3. we focussed on the simplest semi-local operator $e^{i\beta\Phi/2}$;
4. we considered the repulsive regime of the sine-Gordon model;
5. the time t was taken to be large.

It clearly would be interesting to go beyond these restrictions. The treatment of more general vertex operators of the form $e^{i\alpha\Phi}$ appears possible, although the results of Section VI E show that for the local operator $e^{i\beta\Phi}$ the leading term in the linked-cluster expansion vanishes and thus an analysis of higher orders is necessary. Similarly we believe that (a combination of) our methods can be applied with relatively minor extensions to the attractive regime of the sine-Gordon model. To dispense with our other assumptions appears significantly more difficult. Going beyond the leading term in the low-density approximation has proved possible but complicated for the much simpler case of the transverse field Ising chain^{5,16,17}. To do the same for the sine-Gordon case appears to be a daunting task. For example, in the linked-cluster expansion one is faced the problem of late-time singularities at higher orders in the expansion that may no longer simply exponentiate. In the representative state approach, detailed properties of the solutions of the Bethe-Yang equation entering the Lehmann representation (116) will now play a role. Finally, initial states of the form (199) are clearly rather special³⁸. It is not generally known how to express the initial state in terms of Hamiltonian eigenstates after quenching a system parameter. Given the difficulty of this problem, a more fruitful avenue of research would be to follow Ref. [12] and consider special initial states characterized by low entanglement.

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Appendix A: Form-factor axioms

For completeness we state here the used form-factor axioms. We follow Delfino³⁶. The n -particle form factor of an arbitrary operator \mathcal{O} is defined by

$$f_{a_1 \dots a_n}^{\mathcal{O}}(\theta_1, \dots, \theta_n) = \langle 0 | \mathcal{O} | \theta_1, \dots, \theta_n \rangle_{a_1 \dots a_n} = \langle 0 | \mathcal{O} Z_{a_1}^{\dagger}(\theta_1) \dots Z_{a_n}^{\dagger}(\theta_n) | 0 \rangle. \quad (\text{A1})$$

The creation and annihilation operators for solitons and antisolitons are $Z_{\pm}^{\dagger}(\theta)$ and $Z_{\pm}(\theta)$ introduced in Section III. The form-factor axioms read:

1. The form factors $f_{a_1 \dots a_n}^{\mathcal{O}}(\theta_1, \dots, \theta_n)$ are meromorphic functions of θ_N in the physical strip $0 \leq \Im \theta_N \leq 2\pi$. There exist only simple poles in this strip.
2. Scattering axiom:

$$\begin{aligned} f_{a_1 \dots a_i a_{i+1} \dots a_n}^{\mathcal{O}}(\theta_1, \dots, \theta_i, \theta_{i+1}, \dots, \theta_n) \\ = S_{a_i a_{i+1}}^{b_i b_{i+1}}(\theta_i - \theta_{i+1}) f_{a_1 \dots b_{i+1} b_i \dots a_n}^{\mathcal{O}}(\theta_1, \dots, \theta_{i+1}, \theta_i, \dots, \theta_n), \end{aligned}$$

where $S_{a_i a_{i+1}}^{b_i b_{i+1}}(\theta_i - \theta_{i+1})$ is the scattering matrix introduced in Section III.

3. Periodicity axiom:

$$f_{a_1 \dots a_n}^{\mathcal{O}}(\theta_1 + 2\pi i, \theta_2, \dots, \theta_n) = l_{a_1}(\mathcal{O}) f_{a_2 \dots a_n a_1}^{\mathcal{O}}(\theta_2, \dots, \theta_n, \theta_1),$$

where $l_{\pm}(\mathcal{O})$ is the mutual semi-locality factor between the operator \mathcal{O} and the fundamental fields \mathcal{O}_0^{\pm} . In particular, we have for $\mathcal{O} = e^{i\alpha\Phi}$ the factor $l_{\pm}^{\alpha} = e^{\pm i 2\pi \alpha / \beta}$.

4. Lorentz transformations:

$$f_{a_1 \dots a_N}^{\mathcal{O}}(\theta_1 + \Lambda, \dots, \theta_N + \Lambda) = e^{s(\mathcal{O})\Lambda} f_{a_1 \dots a_N}^{\mathcal{O}}(\theta_1, \dots, \theta_N),$$

where $s(\mathcal{O})$ denotes the Lorentz spin of \mathcal{O} . Here we have $s(e^{i\alpha\Phi}) = 0$, i.e. $\mathcal{O} = e^{i\alpha\Phi}$ is a Lorentz scalar.

5. Annihilation pole axiom:

$$\begin{aligned} & \text{Res}[f_{aba_1 \dots a_N}^{\mathcal{O}}(\theta', \theta, \theta_1, \dots, \theta_N), \theta' = \theta + i\pi] \\ &= i C_{ac} f_{b_1 \dots b_N}^{\mathcal{O}}(\theta_1, \dots, \theta_N) \left[\delta_{a_1}^{b_1} \dots \delta_{a_N}^{b_N} \delta_b^c - l_a(\mathcal{O}) S_b^{c_1 b_1}(\theta - \theta_1) S_{c_1 a_2}^{c_2 b_2}(\theta - \theta_2) \dots S_{c_{N-1} a_N}^c(\theta - \theta_N) \right], \end{aligned}$$

with the charge conjugation matrix of the sine-Gordon model given by $C_{ab} = \delta_{a+b,0}$. If there do not exist bound states in the model, i.e. for $\beta^2 \geq 1/2$, these are the only poles of the form factors.

We note in passing that the stated annihilation pole axiom corrects a misprint in Ref. 53 (which had no bearing on the results in said reference).

Appendix B: Hamiltonian eigenstates in the finite volume and the quantum inverse scattering method

In this appendix we summarize some elements of the quantum inverse scattering method⁴¹, which are used to construct Hamiltonian eigenstates in a large, finite volume⁴². We define a *monodromy matrix* by

$$\mathcal{M}_{ab}(\lambda|\{\theta_k\})_{i_1 \dots i_N}^{j_1 \dots j_N} = S_a^{c_1 j_1}(\lambda - \theta_1) \dots S_{c_{N-1} i_N}^b(\lambda - \theta_N). \quad (\text{B1})$$

It can be written as a 2×2 matrix

$$\mathcal{M}(\lambda|\{\theta_k\}) = \begin{pmatrix} \mathcal{A}(\lambda|\{\theta_k\}) & \mathcal{B}(\lambda|\{\theta_k\}) \\ \mathcal{C}(\lambda|\{\theta_k\}) & \mathcal{D}(\lambda|\{\theta_k\}) \end{pmatrix}, \quad (\text{B2})$$

where $\mathcal{A}(\lambda|\{\theta_k\})$, $\mathcal{B}(\lambda|\{\theta_k\})$, $\mathcal{C}(\lambda|\{\theta_k\})$, $\mathcal{D}(\lambda|\{\theta_k\})$ are operators acting on the 2^N dimensional space of scattering states with N particles and fixed rapidities

$$\mathcal{M}_{ab}(\lambda|\{\theta_k\})|\theta_1, \dots, \theta_N\rangle_{i_1 \dots i_N} = \mathcal{M}_{ab}(\lambda|\{\theta_k\})_{i_1 \dots i_N}^{j_1 \dots j_N} |\theta_1, \dots, \theta_N\rangle_{j_1 \dots j_N}. \quad (\text{B3})$$

The rapidities θ_j play the roles of inhomogeneities⁴¹ in the monodromy matrix. The *transfer matrix* is the trace of the monodromy matrix

$$\mathcal{T}(\lambda|\{\theta_k\}) = \sum_a \mathcal{M}_{aa}(\lambda|\{\theta_k\}) = \mathcal{A}(\lambda|\{\theta_k\}) + \mathcal{D}(\lambda|\{\theta_k\}), \quad (\text{B4})$$

As a consequence of the S-matrix being a solution of the Yang-Baxter equation, the monodromy matrices fulfil the relation

$$S_{c_1 c_2}^{b_1 b_2}(\lambda - \mu) \mathcal{M}_{a_1 b_1}(\lambda|\{\theta_k\}) \mathcal{M}_{a_2 b_2}(\mu|\{\theta_k\}) = \mathcal{M}_{d_1 c_1}(\lambda|\{\theta_k\}) \mathcal{M}_{d_2 c_2}(\mu|\{\theta_k\}) S_{a_1 a_2}^{d_1 d_2}(\lambda - \mu), \quad (\text{B5})$$

which implies that the transfer matrices form a commuting family

$$[\mathcal{T}(\lambda|\{\theta_k\}), \mathcal{T}(\mu|\{\theta_k\})] = 0. \quad (\text{B6})$$

Simultaneous eigenstates of all transfer matrices can be constructed by algebraic Bethe Ansatz. Starting point is the “reference state”

$$\Omega(\{\theta_k\}) = |\theta_1, \dots, \theta_N\rangle_{+ \dots +}. \quad (\text{B7})$$

Using the definition of the monodromy matrix, we have

$$\mathcal{M}(\lambda|\{\theta_k\})\Omega(\{\theta_k\}) = \begin{pmatrix} a(\lambda|\{\theta_k\}) & * \\ 0 & d(\lambda|\{\theta_k\}) \end{pmatrix} \Omega(\{\theta_k\}), \quad (\text{B8})$$

where $*$ denotes a non-zero entry not needed in the following, and

$$a(\lambda|\{\theta_k\}) = \prod_{k=1}^N S_0(\lambda - \theta_k), \quad d(\lambda|\{\theta_k\}) = \prod_{k=1}^N S_T(\lambda - \theta_k) \prod_{k=1}^N S_0(\lambda - \theta_k). \quad (\text{B9})$$

Here $S_T(\theta)$ is defined in (21). Taking the trace in (B8) we see that $\Omega(\{\theta_k\})$ is in fact a transfer matrix eigenstate

$$\mathcal{T}(\lambda|\{\theta_k\})\Omega(\{\theta_k\}) = \left[a(\lambda|\{\theta_k\}) + d(\lambda|\{\theta_k\}) \right] \Omega(\{\theta_k\}). \quad (\text{B10})$$

The other eigenvectors of the transfer matrix can be constructed by the Ansatz

$$\Psi(\{\lambda_k\}|\{\theta_k\}) = \left[\prod_{j=1}^r \mathcal{B}(\lambda_j + \frac{i\pi}{2}|\{\theta_k\}) \right] \Omega(\{\theta_k\}), \quad 0 \leq r \leq \frac{N}{2}. \quad (\text{B11})$$

This gives simultaneous eigenvectors of the transfer matrix and the solitonic charge operator Q with a positive eigenvalue of Q ; cf. Subsection B2. States with negative eigenvalues of Q can be obtained acting with the charge conjugation operator C on (B11). Using the commutation relations for $\mathcal{A}(\lambda|\{\theta_k\})$, $\mathcal{B}(\lambda|\{\theta_k\})$, $\mathcal{D}(\lambda|\{\theta_k\})$ that follow from (B5), one observes that (B9) are eigenvectors of the transfer matrix with eigenvalues

$$\Lambda(\lambda, \{\lambda_k\}|\{\theta_i\}) = \left\{ \prod_{k=1}^r S_T(\lambda_k - \lambda + \frac{i\pi}{2})^{-1} + \prod_{k=1}^N S_T(\lambda - \theta_k) \prod_{k=1}^r S_T(\lambda - \lambda_k - \frac{i\pi}{2})^{-1} \right\} \prod_{k=1}^N S_0(\lambda - \theta_k), \quad (\text{B12})$$

if the parameters $\{\lambda_k\}$ satisfy the following set of non-linear algebraic equations

$$\prod_{k=1}^N \frac{\sinh\left(\frac{i\pi}{2\xi} - \frac{\lambda_j - \theta_k}{\xi}\right)}{\sinh\left(\frac{i\pi}{2\xi} + \frac{\lambda_j - \theta_k}{\xi}\right)} = \prod_{\substack{k=1 \\ k \neq j}}^r - \frac{\sinh\left(\frac{i\pi}{\xi} - \frac{\lambda_j - \lambda_k}{\xi}\right)}{\sinh\left(\frac{i\pi}{\xi} - \frac{\lambda_k - \lambda_j}{\xi}\right)}. \quad (\text{B13})$$

Since the equations are $i\pi\xi$ -periodic we impose the restriction $0 \leq \Im \lambda_k \leq \pi\xi \forall k$. Taking the logarithm of (B13) results in equations of the form

$$\mathcal{M}_j(\theta_1, \dots, \theta_N | \lambda_1, \dots, \lambda_r) = 2\pi I_j, \quad I_j \in \mathbb{Z}, \quad j = 1, \dots, r, \quad (\text{B14})$$

where we defined

$$\mathcal{M}_j(\theta_1, \dots, \theta_N | \lambda_1, \dots, \lambda_r) \equiv -i \sum_{k=1}^N \log \frac{\sinh\left(\frac{i\pi}{2\xi} - \frac{\lambda_j - \theta_k}{\xi}\right)}{\sinh\left(\frac{i\pi}{2\xi} + \frac{\lambda_j - \theta_k}{\xi}\right)} + i \sum_{\substack{k=1 \\ k \neq j}}^r \log - \frac{\sinh\left(\frac{i\pi}{\xi} - \frac{\lambda_j - \lambda_k}{\xi}\right)}{\sinh\left(\frac{i\pi}{\xi} - \frac{\lambda_k - \lambda_j}{\xi}\right)}. \quad (\text{B15})$$

1. Periodic boundary conditions

Imposing periodic boundary conditions (56) on the transfer matrix eigenstates gives rise to the following set of “nested” Bethe Ansatz equations

$$e^{i\ell \sinh \theta_i} = \prod_{m=1}^N S_0(\theta_i - \theta_m) \prod_{k=1}^r \frac{\sinh\left(\frac{i\pi}{2\xi} + \frac{\lambda_k - \theta_i}{\xi}\right)}{\sinh\left(\frac{i\pi}{2\xi} - \frac{\lambda_k - \theta_i}{\xi}\right)}, \quad i = 1, \dots, N, \\ \prod_{k=1}^N \frac{\sinh\left(\frac{i\pi}{2\xi} - \frac{\lambda_j - \theta_k}{\xi}\right)}{\sinh\left(\frac{i\pi}{2\xi} + \frac{\lambda_j - \theta_k}{\xi}\right)} = \prod_{\substack{k=1 \\ k \neq j}}^r \frac{\sinh\left(\frac{\lambda_j - \lambda_k - i\pi}{\xi}\right)}{\sinh\left(\frac{\lambda_j - \lambda_k + i\pi}{\xi}\right)}, \quad j = 1, \dots, r. \quad (\text{B16})$$

We recall that $\ell = L\Delta/v$. It is customary to express the logarithmic form of the equations in terms of *counting functions* defined as

$$\ell z(\theta) = \ell \sinh \theta + i \sum_{m=1}^N \ln[S_0(\theta - \theta_m)] + i \sum_{k=1}^r \ln \left[\frac{\sinh\left(\frac{\theta - \lambda_k - \frac{i\pi}{2}}{\xi}\right)}{\sinh\left(\frac{\theta - \lambda_k + \frac{i\pi}{2}}{\xi}\right)} \right],$$

$$\ell y(\lambda) = i \sum_{k=1}^N \ln \left[\frac{\sinh \left(\frac{\lambda - \theta_k}{\xi} - \frac{i\pi}{2\xi} \right)}{\sinh \left(\frac{\lambda - \theta_k}{\xi} + \frac{i\pi}{2\xi} \right)} \right] - i \sum_{k=1}^r \ln \left[\frac{\sinh \left(\frac{\lambda - \lambda_k}{\xi} - \frac{i\pi}{\xi} \right)}{\sinh \left(\frac{\lambda - \lambda_k}{\xi} + \frac{i\pi}{\xi} \right)} \right], \quad (\text{B17})$$

where the branch cuts of the logarithms need to be chosen appropriately. The Bethe-Yang equations then read

$$\begin{aligned} z(\theta_j) &= \frac{2\pi I_j}{\ell}, \quad i = 1, \dots, N, \\ y(\lambda_k) &= \frac{2\pi J_k}{\ell}, \quad k = 1, \dots, r, \end{aligned} \quad (\text{B18})$$

where I_j and J_k are integer or half-odd integer numbers.

2. Topological charge and charge conjugation operators

It is useful to know how the *solitonic charge operator* Q and *charge conjugation operator* C act of the transfer matrix eigenstates. Their respective actions on scattering states (28) are

$$Q |\theta_1, \dots, \theta_N\rangle_{a_1 \dots a_N} = \left\{ \sum_{k=1}^N a_k \right\} |\theta_1, \dots, \theta_N\rangle_{a_1 \dots a_N}, \quad (\text{B19})$$

$$C |\theta_1, \dots, \theta_N\rangle_{a_1 \dots a_N} = |\theta_1, \dots, \theta_N\rangle_{\bar{a}_1 \dots \bar{a}_N}. \quad (\text{B20})$$

It is easily checked that

$$Q \mathcal{M}_{ab}(\lambda|\{\theta_k\}) - \mathcal{M}_{ab}(\lambda|\{\theta_k\})Q = (b - a) \mathcal{M}_{ab}(\lambda|\{\theta_k\}), \quad (\text{B21})$$

$$C \mathcal{M}_{ab}(\lambda|\{\theta_k\})C^{-1} = C \mathcal{M}_{ab}(\lambda|\{\theta_k\})C = \mathcal{M}_{\bar{a}\bar{b}}(\lambda|\{\theta_k\}). \quad (\text{B22})$$

This implies that Q and C commute with the transfer matrix, while

$$[Q, \mathcal{B}(\lambda|\{\theta_k\})] = -2\mathcal{B}(\lambda|\{\theta_k\}), \quad (\text{B23})$$

$$[Q, \mathcal{C}(\lambda|\{\theta_k\})] = 2\mathcal{C}(\lambda|\{\theta_k\}), \quad (\text{B24})$$

$$C \mathcal{B}(\lambda|\{\theta_k\})C = \mathcal{C}(\lambda|\{\theta_k\}), \quad (\text{B25})$$

$$C \mathcal{C}(\lambda|\{\theta_k\})C = \mathcal{B}(\lambda|\{\theta_k\}). \quad (\text{B26})$$

The action of Q on the reference state is

$$Q \Omega(\{\theta_k\}) = N \Omega(\{\theta_k\}), \quad (\text{B27})$$

and hence (B11) are eigenstates of Q with eigenvalue $N - 2r$.

Appendix C: Proof of property (88)

In this appendix we show that the eigenvalues $\Lambda^s(\lambda|\{\theta_k\})$ of the transfer matrix (B6) satisfy the relations

$$\mathcal{K}_{2N}^s(\theta_1, \dots, \theta_N) \Lambda^s(\theta_i|\{\theta_k\}) = \mathcal{K}_{2N}^s(\theta_1, \dots, \theta_N) \Lambda^s(-\theta_i|\{\theta_k\})^{-1}, \quad (\text{C1})$$

where $\{\theta_k\} = \{-\theta_1, \theta_1, \dots, -\theta_N, \theta_N\}$. The proof of (C1) is divided into three main steps.

1. Step I

We first establish the following identity

$$\begin{aligned} & K^{a_1 b_1}(\theta_1) \dots K^{a_N b_N}(\theta_N) \mathcal{T}(\sigma \theta_i | \{-\theta_1, \theta_1, \dots, -\theta_N, \theta_N\})_{a_1 b_1 \dots a_N b_N}^{a'_1 b'_1 \dots a'_N b'_N} \\ &= K^{a_1 b_1}(\theta_1) \dots K^{a_N b_N}(\theta_N) \mathcal{T}(\sigma \theta_i | \{-\theta_N, \theta_N, \dots, -\theta_1, \theta_1\})_{b_N a_N \dots b_1 a_1}^{b'_N a'_N \dots b'_1 a'_1}, \end{aligned} \quad (\text{C2})$$

where $\sigma = \pm 1$ and \mathcal{T} is the transfer matrix

$$\mathcal{T}(\lambda|\{-\theta_1, \theta_1, \dots, -\theta_N, \theta_N\})_{a_1 \dots b_N}^{a'_1 \dots b'_N} = S_{c_{2N}a_1}^{c_1a'_1}(\lambda + \theta_1) S_{c_1b_1}^{c_2b'_1}(\lambda - \theta_1) \dots S_{c_{2N-1}b_N}^{c_{2N}b'_N}(\lambda - \theta_N). \quad (C3)$$

We focus on the case $i = 1$ in (C2), all other cases can be proved analogously. Setting $i = 1$ and $\sigma = -1$, substituting the expression of \mathcal{T} in terms of scattering matrices (C3), and finally using that

$$S_{ab}^{cd}(0) = -\delta_a^d \delta_b^c, \quad (C4)$$

we find that (C2) is reduced to

$$\begin{aligned} K^{a_1b_1}(\theta_1) \dots K^{a_Nb_N}(\theta_N) S_{a_1b_1}^{c_1b'_1}(-2\theta_1) S_{c_1a_2}^{c_2a'_2}(-\theta_1 + \theta_2) \dots S_{c_{2N-1}b_N}^{a'_1b'_N}(-\theta_1 - \theta_N) = \\ K^{a_1b_1}(\theta_1) \dots K^{a_Nb_N}(\theta_N) S_{b_1a_1}^{c_1a'_1}(-2\theta_1) S_{c_1b_N}^{c_2b'_N}(-\theta_1 + \theta_N) \dots S_{c_{2N-1}a_2}^{b'_1a'_2}(-\theta_1 - \theta_2). \end{aligned} \quad (C5)$$

Employing the boundary unitarity property (34) of the K -matrix, the left-hand side of (C5) can be written in the form

$$\begin{aligned} K^{a_1b_1}(\theta_1) \dots K^{a_Nb_N}(\theta_N) S_{a_1b_1}^{c_1b'_1}(-2\theta_1) S_{c_1a_2}^{c_2a'_2}(-\theta_1 + \theta_2) \dots S_{c_{2N-1}b_N}^{a'_1b'_N}(-\theta_1 - \theta_N) = \\ K^{b'_1c_1}(-\theta_1) \dots K^{a_Nb_N}(\theta_N) S_{c_1a_2}^{c_2a'_2}(-\theta_1 + \theta_2) \dots S_{c_{2N-1}b_N}^{a'_1b'_N}(-\theta_1 - \theta_N). \end{aligned} \quad (C6)$$

Now we use the boundary Yang-Baxter equation (33) to rewrite the terms involving θ_1 and θ_2

$$\begin{aligned} K^{b'_1c_1}(-\theta_1) \dots K^{a_Nb_N}(\theta_N) S_{c_1a_2}^{c_2a'_2}(-\theta_1 + \theta_2) \dots S_{c_{2N-1}b_N}^{a'_1b'_N}(-\theta_1 - \theta_N) = \\ K^{c_1c_3}(-\theta_1) \dots K^{a_Nb_N}(\theta_N) S_{c_3a_3}^{c_4a'_3}(-\theta_1 + \theta_3) \dots S_{c_{2N-1}b_N}^{a'_1b'_N}(-\theta_1 - \theta_N) S_{c_1b_2}^{c_2b'_2}(-\theta_1 + \theta_2) S_{c_2a_2}^{b'_1a'_2}(-\theta_1 - \theta_2). \end{aligned} \quad (C7)$$

The index structure on the right-hand side of (C7) is such that we may use the boundary Yang-Baxter equation to rewrite the terms involving θ_1 and θ_3 , then θ_1 and θ_4 and so on. This brings the left-hand side of (C5) to the form

$$\begin{aligned} K^{a_1b_1}(\theta_1) \dots K^{a_Nb_N}(\theta_N) S_{a_1b_1}^{c_1b'_1}(-2\theta_1) S_{c_1a_2}^{c_2a'_2}(-\theta_1 + \theta_2) \dots S_{c_{2N-1}b_N}^{a'_1b'_N}(-\theta_1 - \theta_N) = \\ K^{c_1a'_1}(-\theta_1) \dots K^{a_Nb_N}(\theta_N) S_{c_1b_N}^{c_2b'_N}(-\theta_1 + \theta_N) \dots S_{c_{2N-1}a_2}^{b'_1a'_2}(-\theta_1 - \theta_2). \end{aligned} \quad (C8)$$

Finally we use the reflection equations (34) and parity invariance of the S-matrix

$$S_{ab}^{cd}(\theta) = S_{ba}^{dc}(\theta), \quad (C9)$$

to arrive at

$$K^{a_1b_1}(\theta_1) \dots K^{a_Nb_N}(\theta_N) S_{a_1b_1}^{c_1b'_1}(-2\theta_1) S_{c_1a_2}^{c_2a'_2}(-\theta_1 + \theta_2) \dots S_{c_{2N-1}b_N}^{a'_1b'_N}(-\theta_1 - \theta_N) = \quad (C10)$$

$$K^{a_1b_1}(\theta_1) \dots K^{a_Nb_N}(\theta_N) S_{b_1a_1}^{c_1a'_1}(-2\theta_1) S_{c_1b_N}^{c_2b'_N}(-\theta_1 + \theta_N) \dots S_{c_{2N-1}a_2}^{b'_1a'_2}(-\theta_1 - \theta_2). \quad (C11)$$

This establishes (C5) for $i = 1$ and $\sigma = -1$. The case $\sigma = +1$ is proved in the same way.

2. Step II

Next we establish the following identity

$$[\mathcal{T}(\theta_i|\{-\theta_N, \theta_N, \dots, -\theta_1, \theta_1\})]_{b_N a_N \dots b_1 a_1}^{b'_N a'_N \dots b'_1 a'_1} = [\mathcal{T}(-\theta_i|\{-\theta_1, \theta_1, \dots, -\theta_N, \theta_N\})^{-1}]_{a_1 b_1 \dots a_N b_N}^{a'_1 b'_1 \dots a'_N b'_N}, \quad (C12)$$

which is equivalent to

$$\sum_{a'_1 \dots b'_N} [\mathcal{T}(\theta_i|\{-\theta_N, \dots, \theta_1\})]_{b_N \dots a_1}^{b'_N \dots a'_1} [\mathcal{T}(-\theta_i|\{-\theta_1, \dots, \theta_N\})]_{a'_1 \dots b'_N}^{a''_1 \dots b''_N} = \prod_{i=1}^N \delta_{a_i}^{a''_i} \delta_{b_i}^{b''_i}. \quad (C13)$$

Using the explicit expression of the two transfer matrices

$$[\mathcal{T}(\theta_i|\{-\theta_N, \dots, \theta_1\})]_{b_N \dots a_1}^{b'_N \dots a'_1} = S_{b_i a_{i-1}}^{c_1 a'_{i-1}}(\theta_i + \theta_{i-1}) \dots S_{c_{N-1} a_{i+1}}^{c_N a'_{i+1}}(\theta_i - \theta_{i+1}) S_{c_N b_i}^{a'_i b'_i}(2\theta_i),$$

$$[\mathcal{T}(-\theta_i|\{-\theta_1, \dots, \theta_N\})]_{a'_1 \dots b'_N}^{a'_1 \dots b'_N} = S_{a_i b_i}^{c_1 b'_i}(-2\theta_i) S_{c_1 a_{i+1}}^{c_2 a'_{i+1}}(-\theta_i - \theta_{i+1}) \cdots S_{c_N b_{i-1}}^{a'_i b'_{i-1}}(-\theta_i - \theta_{i-1}), \quad (\text{C14})$$

the left-hand side of Eq. (C13) can be expressed in the form

$$\sum_{a'_1 \dots b'_N} S_{b_i a_{i-1}}^{c_1 a'_{i-1}}(\theta_i + \theta_{i-1}) \cdots S_{c_N b_i}^{a'_i b'_i}(2\theta_i) S_{a'_i b'_i}^{c_1 b'_i}(-2\theta_i) \cdots S_{c_N b'_{i-1}}^{a''_i b'_{i-1}}(-\theta_i - \theta_{i-1}). \quad (\text{C15})$$

The product of the two S-matrices involving θ_i can be simplified using the unitarity condition

$$S_{ab}^{cd}(\theta) S_{cd}^{ef}(-\theta) = \delta_a^e \delta_b^f. \quad (\text{C16})$$

Eq. (C15) then becomes

$$\delta_{b_i}^{b'_i} \sum_{a'_1 \dots b'_N} S_{b_i a_{i-1}}^{c_1 a'_{i-1}}(\theta_i + \theta_{i-1}) \cdots S_{c_N -1 a_{i+1}}^{c_1 a'_{i+1}}(\theta_i - \theta_{i+1}) S_{c_1 a'_{i+1}}^{c_2 a''_{i+1}}(-\theta_i - \theta_{i+1}) \cdots S_{c_N b'_{i-1}}^{a''_i b'_{i-1}}(-\theta_i - \theta_{i-1}). \quad (\text{C17})$$

We observe that now the product of the two S-matrices involving θ_{i+1} can be simplified using (C16). Repeating this procedure for $\theta_{i+2}, \dots, \theta_N, \theta_1, \dots, \theta_{i-1}$ then establishes (C13).

3. Step III

We now substitute (C12) into (C2) to obtain

$$\begin{aligned} & K^{a_1 b_1}(\theta_1) \cdots K^{a_N b_N}(\theta_N) [\mathcal{T}(-\theta_i|\{-\theta_1, \theta_1, \dots, -\theta_N, \theta_N\})]_{a'_1 b'_1 \dots a'_N b'_N}^{a'_1 b'_1 \dots a'_N b'_N} = \\ & = K^{a_1 b_1}(\theta_1) \cdots K^{a_N b_N}(\theta_N) [\mathcal{T}(\theta_i|\{-\theta_1, \theta_1, \dots, -\theta_N, \theta_N\})^{-1}]_{a'_1 b'_1 \dots a'_N b'_N}^{a'_1 b'_1 \dots a'_N b'_N}. \end{aligned} \quad (\text{C18})$$

Combining the eigenvalue equation (51) with the completeness relation (53) we have

$$\Psi_{a'_1 \dots b'_N}^s(\{\theta_k\})^* [\mathcal{T}(-\theta_i|\{-\theta_1, \theta_1, \dots, -\theta_N, \theta_N\})]_{a'_1 b'_1 \dots a'_N b'_N}^{a'_1 b'_1 \dots a'_N b'_N} = \Lambda^s(-\theta_i|\{-\theta_1, \dots, \theta_N\}) \Psi_{a_1 \dots b_N}^s(\{\theta_k\})^*. \quad (\text{C19})$$

Finally we contract both sides of (C18) with $\Psi_{a'_1 \dots b'_N}^s(\{\theta_k\})^*$ and then use (C19) to obtain the desired result

$$\mathcal{K}_{2N}^s(\theta_1, \dots, \theta_N) \Lambda^s(-\theta_i|\{-\theta_1, \dots, \theta_N\}) = \mathcal{K}_{2N}^s(\theta_1, \dots, \theta_N) \Lambda^s(\theta_i|\{-\theta_1, \dots, \theta_N\})^{-1}. \quad (\text{C20})$$

Appendix D: Derivatives of the functions $\bar{Q}_i^s(\theta_1, \dots, \theta_N)$

The functions $\bar{Q}_i^s(\theta_1, \dots, \theta_N)$ are defined by (85) and can be written in the form [cf. (B17)]

$$\bar{Q}_i^s(\theta_1, \dots, \theta_N) = \ell \sinh \theta_i + i \sum_{m=1}^N \ln[S_0(\theta_i - \theta_m) S_0(\theta_i + \theta_m)] + i \sum_{k=1}^r \ln \left[\frac{\sinh \left(\frac{\theta_i - \lambda_k - \frac{i\pi}{2}}{\xi} \right)}{\sinh \left(\frac{\theta_i - \lambda_k + \frac{i\pi}{2}}{\xi} \right)} \right], \quad (\text{D1})$$

where the parameters λ_k are obtained by solving the set of equations

$$2\pi J_m^s = i \sum_{k=1}^N \ln \left[\frac{\sinh \left(\frac{\lambda_m - \theta_k}{\xi} - \frac{i\pi}{2\xi} \right) \sinh \left(\frac{\lambda_m + \theta_k}{\xi} - \frac{i\pi}{2\xi} \right)}{\sinh \left(\frac{\lambda_m - \theta_k}{\xi} + \frac{i\pi}{2\xi} \right) \sinh \left(\frac{\lambda_m + \theta_k}{\xi} + \frac{i\pi}{2\xi} \right)} \right] - i \sum_{k=1}^r \ln \left[\frac{\sinh \left(\frac{\lambda_m - \lambda_k}{\xi} - \frac{i\pi}{\xi} \right)}{\sinh \left(\frac{\lambda_m - \lambda_k}{\xi} + \frac{i\pi}{\xi} \right)} \right], \quad m = 1, \dots, r, \quad (\text{D2})$$

for a given set of (half-odd) integers $\{J_m^s\}$ specifying the polarisation s . We are interested in the situation where

$$\theta_j \approx \begin{cases} -\tilde{\theta}_{N+1-j} & j \leq N \\ \tilde{\theta}_{j-N} & j > N \end{cases}, \quad j = 1, \dots, 2N, \quad (\text{D3})$$

where $\{\tilde{\theta}_j\}$ is the solution of the Bethe-Yang equations corresponding to our representative state, i.e.

$$\bar{Q}_j^s(\tilde{\theta}_1, \dots, \tilde{\theta}_N) = 2\pi \tilde{I}_j, \quad j = 1, \dots, N. \quad (\text{D4})$$

Crucially, for this solution we have

$$\tilde{\theta}_{j+1} - \tilde{\theta}_j = \frac{1}{\ell \rho_\Phi(\theta_j)} + \mathcal{O}(\ell^{-2}), \quad (\text{D5})$$

where $\rho_\Phi(\theta)$ is the root density (115) describing the representative state in the thermodynamic limit. We may use this fact to recast (D2) in the form

$$\frac{2\pi J_m^s}{\ell} = i \int_{-\infty}^{\infty} d\theta \rho_\Phi(\theta) \ln \left[\frac{\sinh\left(\frac{\lambda_m - \theta}{\xi} - \frac{i\pi}{2\xi}\right)}{\sinh\left(\frac{\lambda_m - \theta}{\xi} + \frac{i\pi}{2\xi}\right)} \right] - \frac{i}{\ell} \sum_{k=1}^r \ln \left[\frac{\sinh\left(\frac{\lambda_m - \lambda_k}{\xi} - \frac{i\pi}{\xi}\right)}{\sinh\left(\frac{\lambda_m - \lambda_k}{\xi} + \frac{i\pi}{\xi}\right)} \right] + \mathcal{O}(\ell^{-1}), \quad m = 1, \dots, r. \quad (\text{D6})$$

Denoting the solution to Eqs. (D6) when the $\mathcal{O}(\ell^{-1})$ correction terms are dropped by $\{\lambda_m^{(\infty)}\}$, we conclude that

$$\lambda_m = \lambda_m^{(\infty)} + \mathcal{O}(\ell^{-1}). \quad (\text{D7})$$

Importantly, this conclusion holds for any set $\{\theta_j\}$ that is described by the root density $\rho_\Phi(\theta)$ in the thermodynamic limit. For any such set we then have

$$\bar{Q}_i^s(\theta_1, \dots, \theta_N) = \ell \sinh \theta_i + i \sum_{m=1}^N \ln[S_0(\theta_i - \theta_m)S_0(\theta_i + \theta_m)] + i \sum_{k=1}^r \left(\ln \left[\frac{\sinh\left(\frac{\theta_i - \lambda_k^{(\infty)} - \frac{i\pi}{2}}{\xi}\right)}{\sinh\left(\frac{\theta_i - \lambda_k^{(\infty)} + \frac{i\pi}{2}}{\xi}\right)} \right] + \mathcal{O}(\ell^{-1}) \right). \quad (\text{D8})$$

We conclude that

$$\frac{\partial \bar{Q}_i^s(\theta_1, \dots, \theta_N)}{\partial \theta_j} = \begin{cases} \mathcal{O}(\ell) & i = j, \\ \mathcal{O}(1) & i \neq j. \end{cases} \quad (\text{D9})$$

Appendix E: Contributions from states with $M > N$

In this appendix we use the ideas developed in the Subsection V F to argue that terms with $M > N$ only lead to sub-leading contributions in the Lehmann representation (116). Repeating the reasoning employed to arrive at formula (135), we obtain

$$\begin{aligned} \frac{\text{R}\langle \Psi_0 | e^{i\beta\Phi(t,x)/2} | \Phi_{\tilde{s}} \rangle_{\text{NS}}}{\text{NS}\langle \Psi_0 | \Phi_{\tilde{s}} \rangle_{\text{NS}}} \Big|_{M>N} &= \sum_{s=1}^{2^{2M}} \oint_{\bar{\mathcal{C}}_{\text{tot}}} \prod_{i=1}^M \frac{d\eta_i}{2\pi} E(\eta_1, \dots, \eta_M)_{\tilde{s},s} \\ &+ \sum_{s=1}^{2^{2M}} \sum_{m=1}^N (-1)^m \sum_{1 \leq j_1 < \dots < j_m \leq M} \sum_{i_1, \dots, i_m=1}^N \oint_{\bar{\mathcal{C}}_{i_1, \dots, i_m}^{j_1, \dots, j_m}} \prod_{i=1}^M \frac{d\eta_i}{2\pi} E(\eta_1, \dots, \eta_M)_{\tilde{s},s}. \end{aligned} \quad (\text{E1})$$

Here the multi-contours $\bar{\mathcal{C}}_{\text{tot}}$ and $\bar{\mathcal{C}}_{i_1, \dots, i_m}^{j_1, \dots, j_m}$ are defined analogously to Subsection V F 1, while

$$E(\eta_1, \dots, \eta_M)_{\tilde{s},s} \equiv \frac{\bar{\rho}_N^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N)}{\rho_{2N}^{\tilde{s}}(-\tilde{\theta}_1, \dots, \tilde{\theta}_N)} \frac{\mathcal{K}_{2M}^s(\eta_1, \dots, \eta_M)}{\mathcal{K}_{2N}^{\tilde{s}}(\tilde{\theta}_1, \dots, \tilde{\theta}_N)} \frac{f^{\beta/2}(\eta_M + i\pi, \dots, \tilde{\theta}_N)_{s,\tilde{s}}}{\prod_{k=1}^M \left\{ e^{i\bar{Q}_k^s(\eta_1, \dots, \eta_M)} - 1 \right\}} e^{2i\Delta t \sum_{i=1}^M \cosh \eta_i - \sum_{i=1}^N \cosh \tilde{\theta}_i}. \quad (\text{E2})$$

As we are dealing with a local operator, we expect that significant contributions in the large- N limit can only arise from states with

$$M - N = \mathcal{O}(1). \quad (\text{E3})$$

The leading contribution to (E1) arises from the $\binom{M}{N}$ regions where N of the η_j are integrated around the singularities at $\tilde{\theta}_1, \dots, \tilde{\theta}_N$. In order to obtain an estimate for these contributions we consider the case where $\eta_j \approx \tilde{\theta}_j$, $j = 1, \dots, N$ in more detail. The leading singularity of the form factor is given by

$$f^{\beta/2}(\eta_M + i\pi, \dots, \tilde{\theta}_N)_{s,\tilde{s}} \approx f(\eta_1, \dots, \eta_M)_{\tilde{s},s} \prod_{j=1}^N \frac{1}{(\eta_j - \tilde{\theta}_j)^2}, \quad (\text{E4})$$

where $f(\eta_1, \dots, \eta_M)_{s, \tilde{s}}$ is a regular function. Substituting (E4) back into (E2) and then carrying out the integrals over η_1, \dots, η_N gives a leading contribution of the form

$$\sim \oint_{\mathcal{D}_{1, \dots, N}^{1, \dots, N}} \prod_{i=N+1}^M \frac{d\eta_i}{2\pi} g(\eta_{N+1}, \dots, \eta_M)_{\tilde{s}, s} \prod_{j=1}^N \frac{\partial}{\partial \eta_j} \bigg|_{\eta_j = \tilde{\theta}_j} \left[\prod_{i=1}^M \frac{e^{2i\Delta t \cosh \eta_i}}{e^{i\bar{Q}_i^s(\eta_1, \dots, \eta_M)} - 1} \right] e^{-2i\Delta t \sum_{m=1}^N \cosh \tilde{\theta}_m}. \quad (\text{E5})$$

Here $g(\eta_{N+1}, \dots, \eta_M)_{\tilde{s}, s}$ is a regular function scaling as L^{-N} and $\mathcal{D}_{1, \dots, N}^{1, \dots, N}$ is a multi-contour in \mathbb{C}^{M-N} obtained by removing the first N components from $\bar{\mathcal{C}}_{1, \dots, N}^{1, \dots, N}$. We may now proceed as in Appendix D. In particular, the nested Bethe-Yang equations for solutions $\{\theta_j\}$ such that

$$\theta_k \approx \tilde{\theta}_k, \quad k = 1, \dots, N, \quad (\text{E6})$$

can still be cast in the form (D6). Hence we again have

$$\lambda_m = \lambda_m^{(\infty)} + \mathcal{O}(\ell^{-1}), \quad (\text{E7})$$

which in turn implies that

$$\bar{Q}_i^s(\theta_1, \dots, \theta_M) = \ell \sinh \theta_i + i \sum_{m=1}^M \ln [S_0(\theta_i - \theta_m) S_0(\theta_i - \theta_m)] + i \sum_{k=1}^r \ln \left[\frac{\sinh \left(\frac{\theta_i - \lambda_k^{(\infty)} - \frac{i\pi}{2}}{\xi} \right)}{\sinh \left(\frac{\theta_i - \lambda_k^{(\infty)} + \frac{i\pi}{2}}{\xi} \right)} \right] + \mathcal{O}(1). \quad (\text{E8})$$

Following the same steps as in Subsection V F 1 we then find

$$\begin{aligned} \prod_{j=1}^N \frac{\partial}{\partial \eta_j} \bigg|_{\eta_j = \tilde{\theta}_j} \left[\prod_{i=1}^M \frac{e^{2i\Delta t \cosh \eta_i}}{e^{i\bar{Q}_i^s(\eta_1, \dots, \eta_M)} - 1} \right] e^{-2i\Delta t \sum_{m=1}^N \cosh \tilde{\theta}_m} &\approx \left(\frac{i}{4} \right)^N \prod_{i=1}^N \left\{ \partial_i \bar{Q}_i^s(\tilde{\theta}_1, \dots, \tilde{\theta}_N) - 4\Delta t \sinh \tilde{\theta}_i \right\} \\ &\times \prod_{j=N+1}^M \frac{e^{2i\Delta t \cosh \eta_j}}{e^{i\bar{Q}_j^s(\tilde{\theta}_1, \dots, \tilde{\theta}_N, \eta_{N+1}, \dots, \eta_M)} - 1}. \end{aligned} \quad (\text{E9})$$

The main difference as compared to the $N = M$ case is the presence of additional integrals over $\eta_{N+1}, \dots, \eta_M$. Using that

$$\frac{\partial \bar{Q}_i^s(\theta_1, \dots, \theta_M)}{\partial \theta_j} = \mathcal{O}(1), \quad i \neq j, \quad (\text{E10})$$

we see that for small but fixed $\{\epsilon_i\}$

$$\bar{Q}_i^s(\tilde{\theta}_1, \dots, \tilde{\theta}_N, \theta_{N+1} + i\epsilon_{N+1}, \dots, \theta_M + i\epsilon_M) = \bar{Q}_i^s(\tilde{\theta}_1, \dots, \theta_M) + i\epsilon_i \ell [\cosh \theta_i + \mathcal{O}(1/\ell)], \quad i = N+1, \dots, M. \quad (\text{E11})$$

Formula (E11) implies that for $L \rightarrow \infty$ the parts of the paths composing $\hat{\mathcal{C}}_{\text{tot}}$ below the real axis will give a vanishing contribution. On the remaining parts of the paths we have $\Im \eta_j > 0$, and assuming that we can deform the integration contours a finite distance up into the upper half plane without encountering singularities, we conclude that the resulting contributions are exponentially suppressed in time.

Appendix F: Most singular parts of the form factors

In this appendix we determine the most singular contribution to matrix elements of the form

$$b_N \dots b_1 \langle \theta_N, \dots, \theta_1 | e^{i\beta\Phi(0)/2} | \tilde{\theta}_1, \dots, \tilde{\theta}_N \rangle_{a_1 \dots a_N} = f_{b_N \dots b_1 a_1 \dots a_N}^{\beta/2}(\theta_N + i\pi, \dots, \theta_1 + i\pi, \tilde{\theta}_1, \dots, \tilde{\theta}_N). \quad (\text{F1})$$

We will show by induction that

$$f_{b_N \dots b_1 a_1 \dots a_N}^{\beta/2}(\theta_N + i\pi, \dots, \theta_1 + i\pi, \tilde{\theta}_1, \dots, \tilde{\theta}_N) \bigg|_{\substack{\theta_j \approx \tilde{\theta}_j \\ j=1, \dots, N}} = \mathcal{G}_{\beta/2} \prod_{k=1}^N \frac{2i}{\theta_k - \tilde{\theta}_k} \delta_{a_k, \bar{b}_k} + \text{less singular}. \quad (\text{F2})$$

The case $N = 1$ is an immediate consequence of the annihilation pole axiom (see Appendix A) for the semi-local operator $e^{i\beta\Phi/2}$. We will now assume that (F2) holds and consider

$$f_{b_{N+1}\dots b_1 a_1\dots a_{N+1}}^{\beta/2}(\theta_{N+1} + i\pi, \dots, \theta_1 + i\pi, \tilde{\theta}_1, \dots, \tilde{\theta}_{N+1}) \Big|_{\substack{\theta_j \approx \tilde{\theta}_j \\ j=1, \dots, N+1}}. \quad (\text{F3})$$

Using the periodicity axiom N times, this can be rewritten as

$$(-1)^N f_{b_1 a_1 \dots a_{N+1} b_{N+1} \dots b_2}^{\beta/2}(\theta_1 + i\pi, \tilde{\theta}_1, \dots, \tilde{\theta}_{N+1}, \theta_{N+1} - i\pi, \dots, \theta_2 - i\pi) \Big|_{\substack{\theta_j \approx \tilde{\theta}_j \\ j=1, \dots, N+1}}. \quad (\text{F4})$$

The annihilation pole axiom allows us to extract the leading singularity of this expression for $\theta_1 \approx \tilde{\theta}_1$

$$\begin{aligned} & (-1)^N \frac{i\delta_{c, \bar{b}_1}}{\theta_1 - \tilde{\theta}_1} f_{a'_2 \dots a'_{N+1} b'_{N+1} \dots b'_2}^{\beta/2}(\tilde{\theta}_2, \dots, \tilde{\theta}_{N+1}, \theta_{N+1} - i\pi, \dots, \theta_2 - i\pi) \Big|_{\substack{\theta_j \approx \tilde{\theta}_j \\ j=2, \dots, N+1}} \\ & \times \left[\delta_{a_1}^c \prod_{j=2}^{N+1} \delta_{a_j}^{a'_j} \delta_{b_j}^{b'_j} + S_{a_1 a'_2}^{c_2 a'_2}(\tilde{\theta}_1 - \tilde{\theta}_2) \dots S_{c_N a'_{N+1}}^{c_N a'_{N+1}}(\tilde{\theta}_1 - \tilde{\theta}_{N+1}) S_{c_{N+1} b'_{N+1}}^{d_N b'_{N+1}}(\tilde{\theta}_1 - \theta_{N+1} - i\pi) \dots S_{d_2 b_2}^{c b'_2}(\tilde{\theta}_1 - \theta_2 - i\pi) \right] \quad (\text{F5}) \end{aligned}$$

We now use the periodicity axiom N times to bring the form factor into a form where we can use the induction assumption (F2). The most singular contribution is then given by

$$\begin{aligned} & \frac{i\delta_{c, \bar{b}_1}}{\theta_1 - \tilde{\theta}_1} \prod_{k=2}^{N+1} \frac{2i}{\theta_k - \tilde{\theta}_k} \delta_{a'_k, \bar{b}'_k} \\ & \times \left[\delta_{a_1}^c \prod_{j=2}^{N+1} \delta_{a_j}^{a'_j} \delta_{b_j}^{b'_j} + S_{a_1 a'_2}^{c_2 a'_2}(\tilde{\theta}_1 - \tilde{\theta}_2) \dots S_{c_N a'_{N+1}}^{c_N a'_{N+1}}(\tilde{\theta}_1 - \tilde{\theta}_{N+1}) S_{c_N b_N}^{d_{N-1} b'_N}(\tilde{\theta}_1 - \tilde{\theta}_{N+1} - i\pi) \dots S_{d_2 b_2}^{c b'_2}(\tilde{\theta}_1 - \tilde{\theta}_2 - i\pi) \right]. \quad (\text{F6}) \end{aligned}$$

The product of S-matrices can be simplified by repeatedly using the identity (starting with the two S-matrices involving $\tilde{\theta}_{N+1}$ and then moving outwards in the product)

$$S_{a_1 a'_2}^{c_1 c_2}(\theta) S_{c_1 b_2}^{b_1 \bar{c}_2}(\theta + i\pi) = \delta_{a_1}^{b_1} \delta_{a'_2}^{b_2}. \quad (\text{F7})$$

This completes the induction step.

1. Singularities of form factors with respect to transfer matrix eigenstates

As the most singular contribution to the form factors (F2) is *diagonal* in the indices a_j and b_j , we conclude that the corresponding contribution to form factors involving transfer matrix eigenstates is

$${}^s \langle \theta_N, \dots, \theta_1 | e^{i\beta\Phi(0)/2} | \tilde{\theta}_1, \dots, \tilde{\theta}_N \rangle^{s'} \Big|_{\substack{\theta_j \approx \tilde{\theta}_j \\ j=1, \dots, N}} = \delta_{s, s'} \mathcal{G}_{\beta/2} \prod_{k=1}^N \frac{2i}{\theta_k - \tilde{\theta}_k} + \text{less singular}. \quad (\text{F8})$$

Appendix G: Regularisation of annihilation poles

The correlation functions to be calculated contain matrix elements with incoming and outgoing particles,

$$a_1 \dots a_N \langle \theta'_1, \dots, \theta'_N | \mathcal{O} | \theta_M, \dots, \theta_1 \rangle_{b_M \dots b_1}, \quad (\text{G1})$$

which possess kinematical poles whenever $\theta'_i = \theta_j$ and $a_i = b_j$. These matrix elements can be decomposed into “connected” and “disconnected” contributions. The latter are characterized by the appearance of terms like $\delta(\theta'_i - \theta_j)$, signalling that some of the particles do not encounter the operator \mathcal{O} in the process described by the matrix element.

We follow Smirnov³² to analytically continue form factors. Let $\vec{A} = \{\theta'_1, \dots, \theta'_N\}$ with $\theta'_1 < \theta'_2 < \dots < \theta'_N$ and $\overleftarrow{B} = \{\theta_M, \dots, \theta_1\}$ with $\theta_M > \theta_{M-1} > \dots > \theta_1$ denote two sets of ordered rapidities and introduce the notations

$$Z[\vec{A}]_{a_1 \dots a_N} \equiv Z_{a_1}(\theta'_1) Z_{a_2}(\theta'_2) \dots Z_{a_N}(\theta'_N), \quad (\text{G2})$$

$$Z^\dagger[\overleftarrow{B}]_{b_M \dots b_1} \equiv Z_{b_M}^\dagger(\theta_M) Z_{b_{M-1}}^\dagger(\theta_{M-1}) \dots Z_{b_1}^\dagger(\theta_1). \quad (\text{G3})$$

Now let A_1 and A_2 be a partition of A , i.e. $A = A_1 \cup A_2$, where A_1 contains $n(A_1) = N - k$ rapidities. As a consequence of the Faddeev–Zamolodchikov algebra we have

$$Z[\vec{A}]_{a_1 \dots a_N} = S(\vec{A}|\vec{A}_1)_{a_1 \dots a_N}^{c_1 \dots c_N} Z[\vec{A}_2]_{c_1 \dots c_k} Z[\vec{A}_1]_{c_{k+1} \dots c_N}, \quad (\text{G4})$$

where $S(\vec{A}|\vec{A}_1)$ is the product of two-particle scattering matrices needed to rearrange the order of Faddeev–Zamolodchikov operators in $Z[\vec{A}]$ to arrive at $Z[\vec{A}_2]Z[\vec{A}_1]$. For example, if $\vec{A} = \{\theta'_1, \dots, \theta'_4\}$ and $\vec{A}_1 = \{\theta'_2, \theta'_3\}$ it is given by

$$S(\vec{A}|\vec{A}_1)_{a_1 \dots a_4}^{c_1 \dots c_4} = \delta_{a_1}^{c_4} S_{a_2 b}^{c_2 c_4}(\theta'_2 - \theta'_4) S_{a_3 a_4}^{c_3 b}(\theta'_3 - \theta'_4). \quad (\text{G5})$$

Similarly we have

$$Z^\dagger[\overleftarrow{B}]_{b_M \dots b_1} = Z^\dagger[\overleftarrow{B}_1]_{d_M \dots d_{l+1}} Z^\dagger[\overleftarrow{B}_2]_{d_l \dots d_1} S(\overleftarrow{B}_1|\overleftarrow{B})_{b_M \dots b_1}^{d_M \dots d_1}. \quad (\text{G6})$$

Finally we define

$$\delta(\vec{A}, \overleftarrow{B})_{a_1 \dots a_N}^{b_M \dots b_1} = \delta_{NM} \prod_{j=1}^N 2\pi \delta_{a_j b_j} \delta(\theta'_j - \theta_j). \quad (\text{G7})$$

We are now in a position to analytically continue matrix elements as

$$\begin{aligned} \langle 0 | Z[\vec{A}]_{a_1 \dots a_N} \mathcal{O} Z^\dagger[\overleftarrow{B}]_{b_M \dots b_1} | 0 \rangle &= \sum_{\substack{A=A_1 \cup A_2 \\ B=B_1 \cup B_2}} S(\vec{A}|\vec{A}_1)_{a_1 \dots a_N}^{c_1 \dots c_N} S(\overleftarrow{B}_1|\overleftarrow{B})_{b_M \dots b_1}^{d_M \dots d_1} \delta[\vec{A}_2, \overleftarrow{B}_2]_{d_l \dots d_1}^{c_1 \dots c_k} \\ &\quad \times \langle 0 | Z[\vec{A}_1 + i0]_{c_{k+1} \dots c_N} \mathcal{O} Z^\dagger[\overleftarrow{B}_1]_{d_M \dots d_{l+1}} | 0 \rangle. \end{aligned} \quad (\text{G8})$$

Here the sum is over all possible ways to break the sets A and B into subsets and $\vec{A}_1 + i0$ means that all rapidities in A_1 are slightly moved into the upper half-plane. Similarly, we could choose to analytically continue to the lower half-plane

$$\begin{aligned} \langle 0 | Z[\vec{A}]_{a_1 \dots a_N} \mathcal{O} Z^\dagger[\overleftarrow{B}]_{b_M \dots b_1} | 0 \rangle &= \sum_{\substack{A=A_1 \cup A_2 \\ B=B_1 \cup B_2}} d_{A_2}(\mathcal{O}) S(\vec{A}|\vec{A}_2)_{a_1 \dots a_N}^{c_1 \dots c_N} S(\overleftarrow{B}_2|\overleftarrow{B})_{b_M \dots b_1}^{d_M \dots d_1} \delta[\vec{A}_2, \overleftarrow{B}_2]_{d_l \dots d_1}^{c_1 \dots c_k} \\ &\quad \times \langle 0 | Z[\vec{A}_1 - i0]_{c_{k+1} \dots c_N} \mathcal{O} Z^\dagger[\overleftarrow{B}_1]_{d_M \dots d_{l+1}} | 0 \rangle. \end{aligned} \quad (\text{G9})$$

The factor $d_{A_2}(\mathcal{O})$ is due to a possible semi-locality of the operator \mathcal{O} with respect to the fundamental fields creating the excitations. For the operator $\mathcal{O} = e^{i\alpha\Phi}$ it is given by

$$d_A(e^{i\alpha\Phi}) = \prod_{i=1}^{n(A)} l_{a_i}^\alpha. \quad (\text{G10})$$

The remaining matrix elements in (G8) and (G9) can be evaluated using crossing

$$\begin{aligned} \langle 0 | Z[\vec{A}_1 \pm i0]_{c_{k+1} \dots c_N} \mathcal{O} Z^\dagger[\overleftarrow{B}_1]_{d_M \dots d_{l+1}} | 0 \rangle &= {}_{c_{k+1} \dots c_N} \langle \theta'_{i_{k+1}} \pm i0, \dots, \theta'_{i_N} \pm i0 | \mathcal{O} | \theta_{j_M}, \dots, \theta_{j_{l+1}} \rangle_{d_M \dots d_{l+1}} \\ &= d_{A_1}(\mathcal{O}) C_{c_{k+1} e_{k+1}} \dots C_{c_N e_N} f_{e_{k+1} \dots e_N d_M \dots d_{l+1}}^\mathcal{O}(\theta'_{i_{k+1}} + i\pi \pm i\eta_{i_{k+1}}, \dots, \theta'_{i_N} + i\pi \pm i\eta_{i_N}, \theta_{j_M}, \dots, \theta_{j_{l+1}}), \end{aligned} \quad (\text{G11})$$

where $C_{ab} = \delta_{a+b,0}$ is the charge conjugation matrix and $\eta_i \rightarrow 0^+$. The analytic continuation of general matrix elements (G1) with arbitrary orders of the rapidities can be obtained using the scattering axiom. This regularisation procedure was used previously to study finite-temperature correlations⁴⁹, models with boundaries^{53,54} and quenches in Ising systems^{5,17}.

Appendix H: Infinite-volume regularisation

In this appendix we review the infinite-volume regularisation originally introduced in the study of finite-temperature correlation functions⁴⁹ and recently generalized to the quench problem in the Ising field theory¹⁷. The aim is to exhibit divergences originating in the intertwining of particles with rapidities θ_i and $-\theta_i$ in the initial state. These divergences are a consequence of working in the infinite volume and have to cancel against each other in observables. These cancellations can be made explicit as follows: For each pair of rapidities $\{-\theta_i, \theta_i\}$ in the, say, ket states we introduce an auxiliary real parameter κ_i to shift the rapidities away from the singularities. The resulting expressions have to be understood as generalized functions of the auxiliary variables κ_i . Next we introduce a smooth function $P(\kappa)$ which is strongly peaked around $\kappa = 0$ and satisfies

$$P(0) = L, \quad \int d\kappa P(\kappa) = 1. \quad (\text{H1})$$

Here L can be thought of as the length of the system in a finite-volume regularisation; a possible choice is $P(\kappa) = L\Delta v/[v^2 + (\pi L\Delta\kappa)^2]$.

Appendix I: Two-particle form factors of $\mathcal{O} = e^{i\beta\Phi/2}$

The two-particle form factors of $e^{i\beta\Phi/2}$ have been obtained in Refs. 32, 35, and 51. For the form-factor axioms stated in Appendix A they take the form

$$f_{\pm\mp}^{\beta/2}(\theta_1, \theta_2) = \frac{2i\mathcal{G}_{\beta/2}}{C_1\xi} \frac{\mathcal{G}(\theta_{12})}{\sinh \frac{\theta_{12}-i\pi}{\xi}} e^{\pm(\theta_{12}-i\pi)/(2\xi)}, \quad (\text{I1})$$

where $\theta_{12} = \theta_1 - \theta_2$, $C_1 = \mathcal{G}(i\pi) > 0$ and

$$\mathcal{G}(\theta) = -iC_1 \sinh \frac{\theta}{2} \exp \left(\int_0^\infty \frac{dt \sinh^2(t(1+i\theta/\pi)) \sinh(t(\xi-1))}{t \sinh(2t) \cosh t \sinh(\xi t)} \right) \quad (\text{I2})$$

satisfies $\mathcal{G}(\theta + 2\pi i) = \mathcal{G}(-\theta)$ and $\mathcal{G}(\theta) = S_0(\theta)\mathcal{G}(-\theta)$. An integral representation for the normalisation $\mathcal{G}_{\beta/2} = \langle 0 | e^{i\beta\Phi/2} | 0 \rangle$ was obtained in Ref. 34.

Appendix J: Leading long-time behaviour in $\mathcal{O}(K^4)$

The aim of this appendix is the extraction of the leading long-time behaviour in $\mathcal{O}(K^4)$, i.e. the contribution growing as $\propto t^2$ at late times. To simplify matters we restrict ourselves to Dirichlet-type initial states, $K^{aa}(\xi) = 0$, and do not calculate any terms $\propto \delta(\kappa_i)$, $\propto 1/\kappa_i$ or $\propto t$. The latter implies that we are not able to determine the $\mathcal{O}(K^4)$ corrections to the decay rate (146).

We start with

$$C_{44} = \frac{1}{4} \int_0^\infty \frac{d\theta'_1 d\theta'_2 d\theta_1 d\theta_2}{(2\pi)^4} (K^{a\bar{a}}(\theta'_1))^* (K^{b\bar{b}}(\theta'_2))^* K^{c\bar{c}}(\theta_1) K^{d\bar{d}}(\theta_2) e^{2\Delta i t \sum_i (\cosh \theta'_i - \cosh \theta_i)} \times {}_{\bar{a}\bar{a}\bar{b}\bar{b}} \langle \theta'_1, -\theta'_1, \theta'_2, -\theta'_2 | e^{i\beta\Phi/2} | -\theta_2 + \kappa_2, \theta_2 + \kappa_2, -\theta_1 + \kappa_1, \theta_1 + \kappa_1 \rangle_{d\bar{d}c\bar{c}}. \quad (\text{J1})$$

Since there are two pairs of rapidities in the ket state we have introduced two auxiliary variables $\kappa_{1,2}$. The resulting expressions have to be considered as generalized functions of κ_1 and κ_2 with the final results obtained by multiplication with $P(\kappa_1)P(\kappa_2)$ and integration over both. The regularisation outlined in App. G yields

$${}_{\bar{a}\bar{a}\bar{b}\bar{b}} \langle \theta'_1, -\theta'_1, \theta'_2, -\theta'_2 | e^{i\beta\Phi/2} | -\theta_2 + \kappa_2, \theta_2 + \kappa_2, -\theta_1 + \kappa_1, \theta_1 + \kappa_1 \rangle_{d\bar{d}c\bar{c}} \quad (\text{J2})$$

$$= f_{\bar{a}\bar{a}\bar{b}\bar{b}d\bar{d}c\bar{c}}^{\beta/2}(\theta'_1 + i\pi + i0, -\theta'_1 + i\pi + i0, \theta'_2 + i\pi + i0, -\theta'_2 + i\pi + i0, -\theta_2 + \kappa_2, \theta_2 + \kappa_2, -\theta_1 + \kappa_1, \theta_1 + \kappa_1) \quad (\text{J3})$$

$$-2\pi\delta_a^c \delta(\theta'_1 - \theta_1 - \kappa_1) f_{\bar{a}\bar{b}\bar{d}\bar{d}c\bar{c}}^{\beta/2}(-\theta'_1 + i\pi + i0, \theta'_2 + i\pi + i0, -\theta'_2 + i\pi + i0, -\theta_2 + \kappa_2, \theta_2 + \kappa_2, -\theta_1 + \kappa_1) \quad (\text{J4})$$

$$-2\pi\delta_a^g \delta(\theta'_1 - \theta_2 - \kappa_2) S_{d\bar{c}}^{ef}(\theta_1 + \theta_2) S_{e\bar{c}}^{gh}(\theta_2 - \theta_1) \times f_{\bar{a}\bar{b}\bar{d}\bar{f}h}^{\beta/2}(-\theta'_1 + i\pi + i0, \theta'_2 + i\pi + i0, -\theta'_2 + i\pi + i0, -\theta_2 + \kappa_2, -\theta_1 + \kappa_1, -\theta_1 + \kappa_1) \quad (\text{J5})$$

$$\begin{aligned}
& -2\pi\delta_h^{\bar{e}}\delta(\theta'_2 - \theta_1 - \kappa_1)S_{ab}^{ef}(-\theta'_1 - \theta'_2)S_{af}^{gh}(\theta'_1 - \theta'_2) \\
& \quad \times f_{\bar{g}\bar{e}b\bar{d}\bar{d}\bar{c}}^{\beta/2}(\theta'_1 + i\pi + i0, -\theta'_1 + i\pi + i0, -\theta'_2 + i\pi + i0, -\theta_2 + \kappa_2, \theta_2 + \kappa_2, -\theta_1 + \kappa_1)
\end{aligned} \tag{J6}$$

$$\begin{aligned}
& -2\pi\delta_h^m\delta(\theta'_2 - \theta_2 - \kappa_2)S_{ab}^{ef}(-\theta'_1 - \theta'_2)S_{af}^{gh}(\theta'_1 - \theta'_2)S_{dc}^{ij}(\theta_1 + \theta_2)S_{i\bar{c}}^{mn}(\theta_2 - \theta_1) \\
& \quad \times f_{\bar{g}\bar{e}b\bar{d}j\bar{n}}^{\beta/2}(\theta'_1 + i\pi + i0, -\theta'_1 + i\pi + i0, -\theta'_2 + i\pi + i0, -\theta_2 + \kappa_2, -\theta_1 + \kappa_1, \theta_1 + \kappa_1)
\end{aligned} \tag{J7}$$

$$\begin{aligned}
& -2\pi\delta_e^f\delta(-\theta'_1 + \theta_1 - \kappa_1)S_{aa}^{\bar{e}\bar{e}}(2\theta'_1)S_{cc}^{f\bar{f}}(-2\theta_1) \\
& \quad \times f_{e\bar{b}\bar{b}\bar{d}\bar{d}\bar{f}}^{\beta/2}(\theta'_1 + i\pi + i0, \theta'_2 + i\pi + i0, -\theta'_2 + i\pi + i0, -\theta_2 + \kappa_2, \theta_2 + \kappa_2, \theta_1 + \kappa_1)
\end{aligned} \tag{J8}$$

$$\begin{aligned}
& -2\pi\delta_i^e\delta(-\theta'_1 + \theta_2 - \kappa_2)S_{aa}^{\bar{e}\bar{e}}(2\theta'_1)S_{dd}^{f\bar{f}}(-2\theta_2)S_{fc}^{gh}(\theta_1 - \theta_2)S_{g\bar{c}}^{ij}(-\theta_1 - \theta_2) \\
& \quad \times f_{e\bar{b}\bar{b}\bar{f}h\bar{j}}^{\beta/2}(\theta'_1 + i\pi + i0, \theta'_2 + i\pi + i0, -\theta'_2 + i\pi + i0, \theta_2 + \kappa_2, -\theta_1 + \kappa_1, \theta_1 + \kappa_1)
\end{aligned} \tag{J9}$$

$$\begin{aligned}
& -2\pi\delta_j^f\delta(-\theta'_2 + \theta_1 - \kappa_1)S_{bb}^{\bar{e}\bar{e}}(2\theta'_2)S_{ae}^{gh}(\theta'_2 - \theta'_1)S_{ah}^{ij}(\theta'_1 + \theta'_2)S_{c\bar{c}}^{f\bar{f}}(-2\theta_1) \\
& \quad \times f_{i\bar{g}\bar{e}\bar{d}\bar{d}\bar{f}}^{\beta/2}(\theta'_1 + i\pi + i0, -\theta'_1 + i\pi + i0, \theta'_2 + i\pi + i0, -\theta_2 + \kappa_2, \theta_2 + \kappa_2, \theta_1 + \kappa_1)
\end{aligned} \tag{J10}$$

$$\begin{aligned}
& -2\pi\delta_j^p\delta(-\theta'_2 + \theta_2 - \kappa_2)S_{bb}^{\bar{e}\bar{e}}(2\theta'_2)S_{ae}^{gh}(\theta'_2 - \theta'_1)S_{ah}^{ij}(\theta'_1 + \theta'_2)S_{dd}^{f\bar{f}}(-2\theta_2)S_{fc}^{mn}(\theta_1 - \theta_2)S_{m\bar{c}}^{pq}(-\theta_1 - \theta_2) \\
& \quad \times f_{i\bar{g}\bar{e}\bar{f}n\bar{q}}^{\beta/2}(\theta'_1 + i\pi + i0, -\theta'_1 + i\pi + i0, \theta'_2 + i\pi + i0, \theta_2 + \kappa_2, -\theta_1 + \kappa_1, \theta_1 + \kappa_1)
\end{aligned} \tag{J11}$$

$$\begin{aligned}
& + (2\pi)^2 S_{ab}^{ef}(-\theta'_1 - \theta'_2)S_{dc}^{gh}(\theta_1 + \theta_2)f_{\bar{e}\bar{b}\bar{d}h}^{\beta/2}(-\theta'_1 + i\pi + i0, -\theta'_2 + i\pi + i0, -\theta_2 + \kappa_2, -\theta_1 + \kappa_1) \\
& \quad \times \left[\delta_a^c\delta_f^g\delta(\theta'_1 - \theta_1 - \kappa_1)\delta(\theta'_2 - \theta_2 - \kappa_2) + \delta_a^j\delta_c^{\bar{j}}\delta(\theta'_1 - \theta_2 - \kappa_2)\delta(\theta'_2 - \theta_1 - \kappa_1)S_{gi}^{j\bar{f}}(\theta_2 - \theta'_2) \right]
\end{aligned} \tag{J12}$$

$$\begin{aligned}
& + (2\pi)^2 S_{aa}^{\bar{e}\bar{e}}(2\theta'_1)S_{bb}^{f\bar{f}}(2\theta'_2)S_{ef}^{ij}(\theta'_1 + \theta'_2)S_{cc}^{g\bar{g}}(-2\theta_1)S_{dd}^{h\bar{h}}(-2\theta_2)S_{hg}^{mn}(-\theta_1 - \theta_2) \\
& \quad \times f_{i\bar{f}h\bar{n}}^{\beta/2}(\theta'_1 + i\pi + i0, \theta'_2 + i\pi + i0, \theta_2 + \kappa_2, \theta_1 + \kappa_1) \\
& \quad \times \left[\delta_e^g\delta_j^m\delta(-\theta'_1 + \theta_1 - \kappa_1)\delta(-\theta'_2 + \theta_2 - \kappa_2) + \delta_e^q\delta_g^p\delta(-\theta'_1 + \theta_2 - \kappa_2)\delta(-\theta'_2 + \theta_1 - \kappa_1)S_{mp}^{qj}(\theta'_2 - \theta_2) \right]
\end{aligned} \tag{J13}$$

$$\begin{aligned}
& + (2\pi)^2 \delta_a^c\delta_h^i\delta(\theta'_1 - \theta_1 - \kappa_1)\delta(-\theta'_2 + \theta_2 - \kappa_2)S_{bb}^{\bar{e}\bar{e}}(2\theta'_2)S_{ae}^{gh}(\theta'_2 - \theta'_1)S_{dd}^{f\bar{f}}(-2\theta_2)S_{fc}^{ij}(\theta_1 - \theta_2) \\
& \quad \times f_{\bar{g}\bar{e}\bar{f}j}^{\beta/2}(-\theta'_1 + i\pi + i0, \theta'_2 + i\pi + i0, \theta_2 + \kappa_2, -\theta_1 + \kappa_1)
\end{aligned} \tag{J14}$$

$$\begin{aligned}
& + (2\pi)^2 \delta_a^m\delta_h^n\delta(\theta'_1 - \theta_2 - \kappa_2)\delta(-\theta'_2 + \theta_1 - \kappa_1)S_{bb}^{\bar{e}\bar{e}}(2\theta'_2)S_{ae}^{gh}(\theta'_2 - \theta'_1)S_{c\bar{c}}^{f\bar{f}}(-2\theta_1) \\
& \quad \times S_{d\bar{f}}^{ij}(\theta_2 - \theta_1)S_{if}^{mn}(\theta_1 + \theta_2)f_{\bar{g}\bar{e}\bar{d}j}^{\beta/2}(-\theta'_1 + i\pi + i0, \theta'_2 + i\pi + i0, -\theta_2 + \kappa_2, \theta_1 + \kappa_1)
\end{aligned} \tag{J15}$$

$$\begin{aligned}
& + (2\pi)^2 \delta_j^c\delta_i^m\delta(\theta'_2 - \theta_1 - \kappa_1)\delta(-\theta'_1 + \theta_2 - \kappa_2)S_{aa}^{\bar{e}\bar{e}}(2\theta'_1)S_{eb}^{gh}(\theta'_1 - \theta'_2)S_{eh}^{ij}(-\theta'_1 - \theta'_2)S_{dd}^{f\bar{f}}(-2\theta_2) \\
& \quad \times S_{fc}^{mn}(\theta_1 - \theta_2)f_{\bar{g}\bar{b}\bar{f}n}^{\beta/2}(\theta'_1 + i\pi + i0, -\theta'_2 + i\pi + i0, \theta_2 + \kappa_2, -\theta_1 + \kappa_1)
\end{aligned} \tag{J16}$$

$$\begin{aligned}
& + (2\pi)^2 \delta_e^f\delta_h^i\delta(-\theta'_1 + \theta_1 - \kappa_1)\delta(\theta'_2 - \theta_2 - \kappa_2)S_{aa}^{\bar{e}\bar{e}}(2\theta'_1)S_{eb}^{gh}(\theta'_1 - \theta'_2)S_{c\bar{c}}^{f\bar{f}}(-2\theta_1)S_{d\bar{f}}^{ij}(\theta_2 - \theta_1) \\
& \quad \times f_{\bar{g}\bar{b}\bar{d}j}^{\beta/2}(\theta'_1 + i\pi + i0, -\theta'_2 + i\pi + i0, -\theta_2 + \kappa_2, \theta_1 + \kappa_1)
\end{aligned} \tag{J17}$$

$$+ \dots, \tag{J18}$$

where the dots represent terms that will not lead to contributions $\propto t^2$. An example for such a term is given in Appendix J 4 below. In the following we denote the contribution originating in (J3) by C_{44}^4 , the one originating in (J4)–(J11) by C_{44}^3 and the one originating in (J12)–(J17) by C_{44}^2 . We consider them separately.

1. Leading long-time behaviour of C_{44}^4

We start with the fully connected contribution following from (J3). Similar to the evaluation of C_{22}^2 we shift the contours of integration for θ_1 and θ_2 to the lower-half plane. The leading long-time behaviour originates in the pole

contributions at $\theta_1 = \theta'_1 + \kappa_1 - i0$ and $\theta_2 = \theta'_2 + \kappa_2 - i0$ or $\theta_1 = \theta'_2 + \kappa_1 - i0$ and $\theta_2 = \theta'_1 + \kappa_2 - i0$. Hence we find

$$C_{44}^4 = -\frac{1}{4} \int_0^\infty \frac{d\theta'_1 d\theta'_2}{(2\pi)^2} (K^{a\bar{a}}(\theta'_1))^* (K^{b\bar{b}}(\theta'_2))^* e^{2\Delta i t (\cosh \theta'_1 + \cosh \theta'_2)} \\ \times \left\{ K^{c\bar{c}}(\theta'_2) K^{d\bar{d}}(\theta'_1) e^{-2\Delta i t (\cosh(\theta'_1 + \kappa_1) + \cosh(\theta'_2 + \kappa_2))} \right. \\ \times \text{Res} \left[\text{Res} \left[f_{a\bar{a}b\bar{b}c\bar{c}d\bar{d}}^{\beta/2}(\theta'_1 + i\pi + i0, -\theta'_1 + i\pi + i0, \theta'_2 + i\pi + i0, -\theta'_2 + i\pi + i0, \right. \right. \\ \left. \left. -\theta_2 + \kappa_2, \theta_2 + \kappa_2, -\theta_1 + \kappa_1, \theta_1 + \kappa_1), \theta_1 = \theta'_1 + \kappa_1 - i0 \right], \theta_2 = \theta'_2 + \kappa_2 - i0 \right] \quad (J19)$$

$$+ K^{c\bar{c}}(\theta'_1) K^{d\bar{d}}(\theta'_2) e^{-2\Delta i t (\cosh(\theta'_1 + \kappa_2) + \cosh(\theta'_2 + \kappa_1))} \\ \times \text{Res} \left[\text{Res} \left[f_{a\bar{a}b\bar{b}c\bar{c}d\bar{d}}^{\beta/2}(\theta'_1 + i\pi + i0, -\theta'_1 + i\pi + i0, \theta'_2 + i\pi + i0, -\theta'_2 + i\pi + i0, \right. \right. \\ \left. \left. -\theta_2 + \kappa_2, \theta_2 + \kappa_2, -\theta_1 + \kappa_1, \theta_1 + \kappa_1), \theta_1 = \theta'_2 + \kappa_1 - i0 \right], \theta_2 = \theta'_1 + \kappa_2 - i0 \right] \left. \right\} \quad (J20)$$

+ ...,

where the dots represent terms which grow at most as t . On the other hand, the leading time dependence originating from the exponentials has the form $\propto \kappa_1 \kappa_2 t^2$. Since we are interested in this t^2 -behaviour only, we thus have to extract the contributions $\propto 1/(\kappa_1 \kappa_2)$ from the double residue. This is done by applying the form-factor axioms stated in Appendix A, resulting in

$$C_{44}^4 = C_{44}^{4,1} + C_{44}^{4,2} + \dots, \quad (J21)$$

$$C_{44}^{4,1} = \frac{\mathcal{G}_{\beta/2} \Delta^2 t^2}{\pi^2} \int_0^\infty d\theta'_1 d\theta'_2 |K^{a\bar{a}}(\theta'_1)|^2 |K^{b\bar{b}}(\theta'_2)|^2 \sinh \theta'_1 \sinh \theta'_2, \quad (J22)$$

$$C_{44}^{4,2} = \frac{\mathcal{G}_{\beta/2} \Delta^2 t^2}{4\pi^2} \int_0^\infty d\theta'_1 d\theta'_2 (K^{a\bar{a}}(\theta'_1))^* (K^{b\bar{b}}(\theta'_2))^* K^{c\bar{c}}(\theta'_1) K^{d\bar{d}}(\theta'_2) \sinh \theta'_1 \sinh \theta'_2 \\ \times \left\{ S_{gf}^{\bar{d}c}(\theta'_1 + \theta'_2) S_{ab}^{\bar{e}k}(-\theta'_1 - \theta'_2) S_{be}^{\bar{g}c}(\theta'_2 - \theta'_1) S_{fd}^{ak}(\theta'_1 - \theta'_2) \right. \quad (J23)$$

$$+ S_{a\bar{a}}^{m\bar{m}}(2\theta'_1) S_{ef}^{pq}(-2\theta'_1) S_{gf}^{\bar{d}c}(\theta'_1 + \theta'_2) S_{be}^{\bar{g}c}(\theta'_2 - \theta'_1) S_{pd}^{mk}(-\theta'_1 - \theta'_2) S_{qk}^{\bar{m}b}(\theta'_1 - \theta'_2) \quad (J24)$$

$$+ S_{bb}^{e\bar{e}}(2\theta'_2) S_{d\bar{d}}^{f\bar{f}}(-2\theta'_2) S_{a\bar{e}}^{gh}(\theta'_2 - \theta'_1) S_{ah}^{i\bar{f}}(\theta'_1 + \theta'_2) S_{ge}^{\bar{c}k}(-\theta'_1 - \theta'_2) S_{cf}^{ik}(\theta'_1 - \theta'_2) \quad (J25)$$

$$+ S_{bb}^{e\bar{e}}(2\theta'_2) S_{d\bar{d}}^{f\bar{f}}(-2\theta'_2) S_{ig}^{mn}(2\theta'_1) S_{c\bar{c}}^{k\bar{k}}(-2\theta'_1) S_{a\bar{e}}^{gh}(\theta'_2 - \theta'_1) S_{ah}^{i\bar{f}}(\theta'_1 + \theta'_2) S_{kf}^{\bar{n}p}(-\theta'_1 - \theta'_2) S_{k\bar{p}}^{me}(\theta'_1 - \theta'_2) \left. \right\} \quad (J26)$$

$$\equiv C_{44}^{4,2,a} + C_{44}^{4,2,b} + C_{44}^{4,2,c} + C_{44}^{4,2,d}, \quad (J27)$$

where the dots in (J21) represent terms that grow at most as t at large times as well as terms $\propto \delta(\kappa_i)$ or $\propto 1/\kappa_i$.

2. Leading long-time behaviour of C_{44}^3

We now turn to the disconnected pieces originating in (J4)–(J11). If we label the resulting terms consecutively from $C_{44}^{3,1}$ to $C_{44}^{3,8}$, we find after straightforward evaluation

$$C_{44}^{3,1} = C_{44}^{3,4} = -C_{44}^{3,5} = -C_{44}^{3,8} = \frac{1}{2} C_{44}^{4,1}, \quad (J28)$$

$$C_{44}^{3,2} = C_{44}^{4,2,a} + C_{44}^{4,2,b}, \quad (J29)$$

$$C_{44}^{3,3} = C_{44}^{4,2,a} + C_{44}^{4,2,c}, \quad (J30)$$

$$C_{44}^{3,6} = -C_{44}^{4,2,b} - C_{44}^{4,2,d}, \quad (J31)$$

$$C_{44}^{3,7} = -C_{44}^{4,2,c} - C_{44}^{4,2,d}. \quad (J32)$$

Thus in total we find

$$C_{44}^3 = \sum_{i=1}^8 C_{44}^{3,i} = 2C_{44}^{4,2,a} - 2C_{44}^{4,2,d}. \quad (J33)$$

3. Leading long-time behaviour of C_{44}^2

Similarly we label the terms originating from (J12)–(J17) by $C_{44}^{2,1}$ to $C_{44}^{2,6}$. Straightforward evaluation of the leading long-time behaviour then yields

$$C_{44}^{2,1} = \frac{1}{4}C_{44}^{4,1} + C_{44}^{4,2,a}, \quad (\text{J34})$$

$$C_{44}^{2,2} = \frac{1}{4}C_{44}^{4,1} + C_{44}^{4,2,d}, \quad (\text{J35})$$

$$C_{44}^{2,3} = C_{44}^{2,6} = -\frac{1}{4}C_{44}^{4,1}, \quad (\text{J36})$$

$$C_{44}^{2,4} = -C_{44}^{4,2,c}, \quad (\text{J37})$$

$$C_{44}^{2,5} = -C_{44}^{4,2,b}. \quad (\text{J38})$$

Hence we find in total

$$C_{44}^2 = \sum_{i=1}^6 C_{44}^{2,i} = C_{44}^{4,2,a} - C_{44}^{4,2,b} - C_{44}^{4,2,c} + C_{44}^{4,2,d}. \quad (\text{J39})$$

4. Example for a contribution not leading to t^2 -behaviour

As an example for a term not leading to a contribution $\propto t^2$ we consider another disconnected piece appearing in the regularisation of (J2), which is given by

$$(2\pi)^2 \delta_a^c \delta_a^f \delta(\theta'_1 - \theta_1 - \kappa_1) \delta(-\theta'_1 + \theta_2 - \kappa_2) S_{d\bar{d}}^{e\bar{e}}(-2\theta_2) S_{ec}^{fg}(\theta_1 - \theta_2) f_{b\bar{b}e\bar{g}}^{\beta/2}(\theta'_2 + i\pi + i0, -\theta'_2 + i\pi + i0, \theta_2 + \kappa_2, -\theta_1 + \kappa_1). \quad (\text{J40})$$

This yields

$$C_{44}^{2,7} \equiv \frac{1}{4} \int_0^\infty \frac{d\theta'_1 d\theta'_2}{(2\pi)^2} |K^{a\bar{a}}(\theta'_1)|^2 (K^{b\bar{b}}(\theta'_2))^* K^{d\bar{d}}(\theta'_1) e^{2\Delta i t (\cosh \theta'_1 + \cosh \theta'_2 - \cosh(\theta'_1 - \kappa_1) - \cosh(\theta'_1 + \kappa_2))} \quad (\text{J41})$$

$$\times S_{d\bar{d}}^{e\bar{e}}(-2\theta'_1) S_{ea}^{af}(-\kappa_1 - \kappa_2) f_{b\bar{b}e\bar{f}}^{\beta/2}(\theta'_2 + i\pi + i0, -\theta'_2 + i\pi + i0, \theta'_1 + 2\kappa_2, -\theta'_1 + 2\kappa_1). \quad (\text{J42})$$

Now shifting the θ'_2 -contour to the upper half plane we pick up a pole at $\theta'_2 = \theta'_1 - 2\kappa_1 + i0$. Evaluating the corresponding residue and extracting the contribution $\propto t^2$ we find

$$C_{44}^{2,7} \propto \frac{\kappa_1 \kappa_2}{2\kappa_1 + 2\kappa_2 - i0} \Delta^2 t^2 + \dots \quad (\text{J43})$$

However, multiplying this with $P(\kappa_1) P(\kappa_2)$ and performing the integrations over $\kappa_{1,2}$ we find

$$\int d\kappa_1 d\kappa_2 \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2 - i0} P(\kappa_1) P(\kappa_2) \propto \frac{1}{L} \rightarrow 0. \quad (\text{J44})$$

Hence $C_{44}^{2,7}$ contains no contribution $\propto t^2$.

5. Leading long-time behaviour of C_{44}

Taking together (J21), (J33), and (J39) we find for the leading long-time behaviour in $\mathcal{O}(K^4)$:

$$C_{44} = C_{44}^1 + 4 C_{44}^{4,2,a} + \dots \quad (\text{J45})$$

$$= \frac{\mathcal{G}_{\beta/2} \Delta^2 t^2}{\pi^2} \int_0^\infty d\theta_1 d\theta_2 |K^{a\bar{a}}(\theta_1)|^2 |K^{b\bar{b}}(\theta_2)|^2 \sinh \theta_1 \sinh \theta_2 \quad (\text{J46})$$

$$+ \frac{\mathcal{G}_{\beta/2} \Delta^2 t^2}{\pi^2} \int_0^\infty d\theta_1 d\theta_2 (K^{a\bar{a}}(\theta_1))^* (K^{b\bar{b}}(\theta_2))^* K^{c\bar{c}}(\theta_1) K^{d\bar{d}}(\theta_2) T_{abcd}(\theta_1, \theta_2) \sinh \theta_1 \sinh \theta_2, \quad (\text{J47})$$

$$T_{abcd}(\theta_1, \theta_2) = S_{ab}^{ef}(-\theta_1 - \theta_2) S_{cd}^{gh}(\theta_1 + \theta_2) S_{af}^{gd}(\theta_1 - \theta_2) S_{ch}^{eb}(\theta_2 - \theta_1), \quad (\text{J48})$$

where the dots represent terms that grow at most as t at large times as well as terms $\propto \delta(\kappa_i)$ that have to cancel against the normalisation of the initial state. The term (J47) can be further simplified. We multiply the reflection equation (33) by $S_{b_2 b_1}^{d_2 d_1}(\theta_2 - \theta_1)$ and sum over $b_{1,2}$, followed by multiplication with $S_{a_2 d_1}^{e_2 e_1}(-\theta_1 - \theta_2)$ and summing over a_2 and d_1 . This yields

$$K^{a_1 e_1}(\theta_1) K^{e_2 d_2}(\theta_2) = K^{c_1 b_1}(\theta_1) K^{c_2 c_3}(\theta_2) S_{c_3 c_1}^{b_2 c_4}(\theta_1 + \theta_2) S_{c_2 c_4}^{a_2 a_1}(\theta_1 - \theta_2) S_{b_2 b_1}^{d_2 d_1}(\theta_2 - \theta_1) S_{a_2 d_1}^{e_2 e_1}(-\theta_1 - \theta_2). \quad (\text{J49})$$

Now considering initial states satisfying $K^{ab}(\theta) = K^{a\bar{a}}(\theta) \delta_a^{\bar{b}}$ we find

$$K^{a_1 \bar{a}_1}(\theta_1) K^{e_2 \bar{e}_2}(\theta_2) \delta_{\bar{a}_1}^{e_1} \delta_{\bar{e}_2}^{d_2} = K^{c_1 \bar{c}_1}(\theta_1) K^{c_2 \bar{c}_2}(\theta_2) S_{\bar{c}_2 c_1}^{b_2 c_4}(\theta_1 + \theta_2) S_{c_2 c_4}^{a_2 a_1}(\theta_1 - \theta_2) S_{b_2 \bar{c}_1}^{d_2 d_1}(\theta_2 - \theta_1) S_{a_2 d_1}^{e_2 e_1}(-\theta_1 - \theta_2). \quad (\text{J50})$$

This equation in particular holds for $e_1 = \bar{a}_1$ and $d_2 = \bar{e}_2$, thus after relabeling the indices and using the properties (26) we arrive at

$$K^{a\bar{a}}(\theta_1) K^{b\bar{b}}(\theta_2) = K^{c\bar{c}}(\theta_1) K^{d\bar{d}}(\theta_2) T_{abcd}(\theta_1, \theta_2). \quad (\text{J51})$$

Hence we find for the leading long-time behaviour in $\mathcal{O}(K^2)$

$$C_{44} = \frac{2\mathcal{G}_{\beta/2} \Delta^2 t^2}{\pi^2} \int_0^\infty d\theta_1 d\theta_2 |K^{a\bar{a}}(\theta_1)|^2 |K^{b\bar{b}}(\theta_2)|^2 \sinh \theta_1 \sinh \theta_2 + \dots, \quad (\text{J52})$$

i.e. the result stated in (189).

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